# LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

# Lecture 5: Determinant and inverses of block matrices

- We recall: Fix $m, n \in \mathbb{N}$. Consider a vector $z \in \mathbb{R}^{m+n}$:

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+n} \end{pmatrix}.$$

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$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+n} \end{pmatrix}.$$

- We can view of the first $m$-coordinates of $z$ as forming a vector in $\mathbb{R}^m$ and the remaining $n$-coordinates as forming a vector in $\mathbb{R}^n$.
- So we write

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$x = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}, \quad y = \begin{pmatrix} z_{m+1} \\ z_{m+2} \\ \vdots \\ z_{m+n} \end{pmatrix}.$$

# Continuation

- Conversely, given any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we get a vector $z \in \mathbb{R}^{m+n}$ as

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- Conversely, given any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we get a vector $z \in \mathbb{R}^{m+n}$ as
$$z = \begin{pmatrix} x \\ y \end{pmatrix}.$$
- So in a way, we can think of $\mathbb{R}^{m+n}$ as constructed out of $\mathbb{R}^m$ and $\mathbb{R}^n$. We say that $\mathbb{R}^{m+n}$ is direct sum of $\mathbb{R}^m$ and $\mathbb{R}^n$.

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- We partition $P$ as
$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$
where $A_{m\times m}, B_{m\times n}, C_{n\times m}, D_{n\times n}$ are given by
$$A = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix}, \quad B = \begin{bmatrix} p_{1(m+1)} & \cdots & p_{1(m+n)} \\ \vdots & \ddots & \vdots \\ p_{m(m+1)} & \cdots & p_{m(m+n)} \end{bmatrix}.$$

# Continuation

- $$C = \begin{bmatrix} p_{(m+1)1} & \cdots & p_{(m+1)m} \\ \vdots & \ddots & \vdots \\ p_{(m+n)1} & \cdots & p_{(m+n)(m)} \end{bmatrix},$$

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- $$D = \begin{bmatrix} p_{(m+1)(m+1)} & \cdots & p_{(m+1)(m+n)} \\ \vdots & \ddots & \vdots \\ p_{(m+n)(m+1)} & \cdots & p_{(m+n)(m+n)} \end{bmatrix}$$

# The action of partitioned matrices on vectors

- Under notation as above, with

$$Pz = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}.$$

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- Note that $A : \mathbb{R}^m \to \mathbb{R}^m$, $B : \mathbb{R}^n \to \mathbb{R}^m$, $C : \mathbb{R}^m \to \mathbb{R}^n$ and $D : \mathbb{R}^n \to \mathbb{R}^n$.

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- In other words, the multiplication is like the usual matrix multiplication.
- Proof. The proof is by direct multiplication.

# Continuation

- For instance, for $1 \leq i, j \leq m$,

$$\begin{aligned}(PQ)_{ij} &= \sum_{k=1}^{m+n} p_{ik} q_{kj} = \sum_{k=1}^{m} p_{ik} q_{kj} + \sum_{k=m+1}^{m+n} p_{ik} q_{kj} \\ &= (AE)_{ij} + (BG)_{ij}\end{aligned}$$

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- More generally, if $P = [A_{ij}], \quad Q = [B_{kl}]$ are partitioned matrices, with matching orders, then $PQ$ is a partitioned matrix $[C_{ij}]$ with

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- Here, for the matrix multiplication to be meaningful, it is necessary that for fixed $i, k, j$, if the order of $A_{ik}$ is $a \times b$ then the order of $B_{kj}$ should be $b \times c$ for some $c$. This is what we mean by 'matching orders'.

# Determinants of block upper triangular matrices

- Theorem 4.4: Consider a block upper triangular matrix

$$P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where $A, D$ are square matrices. Then

$$\det(P) = \det(A). \det(D).$$

# Inverses of $2 \times 2$ upper triangular matrices.

- Theorem 4.5: Consider a block upper triangular matrix

$$P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where $A, D$ are square matrices. Then $P$ is invertible if and only if $A$ and $D$ are invertible and in such a case,

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- From the formula $\det(P) = \det(A).\det(D)$, we know that if $P$ is invertible, then $\det(A)$ and $\det(D)$ are non-zero and hence $A, D$ are invertible.

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- From the formula $\det(P) = \det(A).\det(D)$, we know that if $P$ is invertible, then $\det(A)$ and $\det(D)$ are non-zero and hence $A, D$ are invertible.
- The formula for the inverse can be confirmed by verifying:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} . \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

# A special case

- Corollary 4.6: For any matrix $B$,

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^n = \begin{bmatrix} I & nB \\ 0 & I \end{bmatrix}$$

for every $n \in \mathbb{Z}$.

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- Proof: The result is clear for $n = 0, 1$. Now verify the formula for $n \in \mathbb{N}$ by induction. Taking inverses we have the result for all $n \in \mathbb{Z}$.

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- This is actually a consequence of

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B + C \\ 0 & I \end{bmatrix}.$$

The matrix product becomes simple addition here.

# A factorization theorem for $2 \times 2$ block matrices

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- (ii) If $A$ is invertible, then
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- Remark The terms $A - BD^{-1}C$ and $D - CA^{-1}B$ appearing above are known as Schur Complements and they appear in various block matrix computations.

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- Remark The terms $A - BD^{-1}C$ and $D - CA^{-1}B$ appearing above are known as Schur Complements and they appear in various block matrix computations.
- Here $A$ and $D$ need not be of same order.

# Continuation

- Proof. By direct computation:

$$\begin{aligned}
&\begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \\
= &\begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \\
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- This proves (i). Similarly (ii) follows by multiplication. ■

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- Proof. Clear from the factorization result and the fact that the determinant of a triangular block matrix is the product of determinants of diagonal blocks.

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- (ii) If $A$ is invertible, and $T := D - CA^{-1}B$ is invertible, then $P$ is invertible and

$$P^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BT^{-1}CA^{-1} & -A^{-1}BT^{-1} \\ -T^{-1}CA^{-1} & T^{-1} \end{bmatrix}.$$

# Some special cases

- Theorem 5.4: Suppose

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D$ are square matrices of same sizes and $C, D$ commute ($CD = DC$). Then

$$\det(P) = \det(AD - BC).$$

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- Theorem 5.5: Suppose

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- Exercise: Prove these theorems.

# Matrix tricks

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where $B, C$ are $n \times n$ square matrices and either $B$ or $C$ is invertible.

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- Now

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} C & D \\ A & B \end{bmatrix}$$

# Continuation

- Therefore,

$$\det(P) = (-1)^n . \det\left(\begin{bmatrix} C & D \\ A & B \end{bmatrix}\right)$$

which can be computed using the formulae derived earlier.

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- Hence there exists $\epsilon > 0$ such that $f(t) \neq 0$ for $t \in (-\epsilon, \epsilon)\setminus\{0\}$.

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- Then $f$ is a polynomial in $t$. So it has finite number of zeros.
- Hence there exists $\epsilon > 0$ such that $f(t) \neq 0$ for $t \in (-\epsilon, \epsilon) \backslash \{0\}$.
- Therefore $A + tI$ is invertible for $t \in (-\epsilon, \epsilon) \backslash \{0\}$.
- So using results proved earlier we can compute the determinant of $P_t$ for $t \in (-\epsilon, +\epsilon) \backslash \{0\}$.

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- For $t \in \mathbb{R}$ consider
$$P_t = P + tI = \left[ \begin{array}{cc} A + tI & B \\ C & D + tI \end{array} \right]$$
- Consider $f(t) = \det(A + tI)$ for $t \in \mathbb{R}$.
- Then $f$ is a polynomial in $t$. So it has finite number of zeros.
- Hence there exists $\epsilon > 0$ such that $f(t) \neq 0$ for $t \in (-\epsilon, \epsilon) \backslash \{0\}$.
- Therefore $A + tI$ is invertible for $t \in (-\epsilon, \epsilon) \backslash \{0\}$.
- So using results proved earlier we can compute the determinant of $P_t$ for $t \in (-\epsilon, +\epsilon) \backslash \{0\}$.
- Taking the limit as $t$ tends to 0, we get the determinant of $P$.

# Another trick

- Suppose we want to compute the determinant of
$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
where $A, D$ are square matrices, but not invertible.
- For $t \in \mathbb{R}$ consider
$$P_t = P + tI = \begin{bmatrix} A + tI & B \\ C & D + tI \end{bmatrix}$$
- Consider $f(t) = \det(A + tI)$ for $t \in \mathbb{R}$.
- Then $f$ is a polynomial in $t$. So it has finite number of zeros.
- Hence there exists $\epsilon > 0$ such that $f(t) \neq 0$ for $t \in (-\epsilon, \epsilon)\backslash\{0\}$.
- Therefore $A + tI$ is invertible for $t \in (-\epsilon, \epsilon)\backslash\{0\}$.
- So using results proved earlier we can compute the determinant of $P_t$ for $t \in (-\epsilon, +\epsilon)\backslash\{0\}$.
- Taking the limit as $t$ tends to 0, we get the determinant of $P$.
- END OF LECTURE 5.