LINEAR ALGEBRA -II

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Lecture 6: Cauchy Binet formula

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- Question: How can we compute the determinant of AB using A and B?
- This question is answered by Cauchy-Binet formula.

Illustration

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- Then
$$\det(AB) = \det(A_1 B_1) + \det(A_2 B_2) + \det(A_3 B_3),$$
- where A_j, B_j's are 2×2 matrices formed by choosing columns of A and respective rows of B:
$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 & 0 \\ 5 & 7 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 7 & 8 \\ 0 & 1 \end{bmatrix}.$$

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- Suppose $B = [b_{jk}]_{1 \leq j \leq n, 1 \leq k \leq m}$ so that B is an $n \times m$ matrix.
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- So if $1 \leq j_1, j_2, \ldots, j_m \leq n$, we take

$$B(j_1, j_2, \ldots, j_m | 1, 2, \ldots, m) = \begin{bmatrix} b_{j_1 1} & b_{j_1 2} & \ldots & b_{j_1 m} \\ b_{j_2 1} & b_{j_2 2} & \ldots & b_{j_2 m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{j_m 1} & b_{j_m 2} & \ldots & b_{j_m m} \end{bmatrix}.$$

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- The notation indicates that the rows chosen are $j_1, j_2, \ldots, j_m$, and columns chosen are $1, 2, \ldots, m$.

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- Similarly if $A = [a_{ij}]_{1 \leq i \leq m; 1 \leq j \leq n}$ is an $m \times n$ matrix with $m \leq n$, we form square matrices by choosing m-columns of A.

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$$A(1, 2, \ldots, m | j_1, j_2, \ldots, j_m) := \begin{bmatrix} a_{1j_1} & a_{1j_2} & \ldots & a_{1j_m} \\ a_{2j_1} & a_{2j_2} & \ldots & a_{2j_m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{mj_1} & a_{mj_2} & \ldots & a_{mj_m} \end{bmatrix}.$$

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- Here all rows are chosen and columns $j_1, j_2, \ldots, j_m$ are chosen to get a square matrix.

A Lemma

- Lemma 6.1: For any $n \times m$ matrix B,

$$\det B(j_1, j_2, \ldots, j_m | 1, 2, \ldots, m) = 0$$

if $j_1, \ldots, j_m$ are not distinct. If $j_1, j_2, \ldots, j_m$ are distinct, then

$$\begin{aligned} & \det B(j_1, j_2, \ldots, j_m | 1, \ldots, m) \\ = \; & \epsilon(\tau) \det B(j_{\tau(1)}, j_{\tau(2)}, \ldots, j_{\tau(m)} | 1, \ldots, m) \end{aligned}$$

where $\tau \in S_m$ is the permutation such that $j_{\tau(1)} < j_{\tau(2)} < \cdots < j_{\tau(m)}$.

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- Proof. Follows from the basic properties of the determinant.

Cauchy Binet formula

- Theorem 6.2: Suppose A, B are $m \times n$ and $n \times m$ matrices with $m \leq n$ and $C = AB$. Then $\det(C) =$

$$\sum_{1 \leq j_1 < \cdots < j_m \leq n} \det(A(1, \ldots, m | j_1, \ldots, j_m)). \det(B(j_1, \ldots, j_m | 1, \ldots, m)).$$

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- Note that there are $\begin{pmatrix} n \\ m \end{pmatrix}$ terms in this summation.
- Proof. We have

$$\begin{aligned} & \det(AB) \\ = & \sum_{\sigma \in S_m} \epsilon(\sigma)(AB)_{1\sigma(1)}(AB)_{2\sigma(2)} \ldots (AB)_{m\sigma(m)} \\ = & \sum_{\sigma \in S_m} \epsilon(\sigma)(\sum_{j=1}^{n} a_{1j} b_{j\sigma(1)})(\sum_{j=1}^{n} a_{2j} b_{j\sigma(2)}) \cdots (\sum_{j=1}^{n} a_{mj} b_{j\sigma(m)}) \end{aligned}$$

Continuation

$$= \sum_{j_1,j_2,\dots,j_m=1}^{n} a_{1j_1}\dots a_{mj_m} \sum_{\sigma\in S_m} \epsilon(\sigma) b_{j_1\sigma(1)} b_{j_2\sigma(2)} \cdots b_{j_m\sigma(m)}$$

$$= \sum_{j_1,j_2,\dots,j_m=1}^{n} a_{1j_1}\dots a_{mj_m} \det B(j_1,\dots,j_m|1,\dots m)$$

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$$= \sum_{1\le j_1<\cdots<j_m\le n} \det A(1,\dots,m|j_1,\dots,j_m) \det B(j_1,\dots,j_m|1,\dots,m)$$

This is what we wanted to prove. ■

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- END OF LECTURE 6.