LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 7: Inner product spaces

- You have studied vector spaces. Now we introduce a new structure on them called 'inner product'.

Lecture 7: Inner product spaces

- You have studied vector spaces. Now we introduce a new structure on them called 'inner product'.
- This abstractly captures the notions of 'length' and 'angle'.

Lecture 7: Inner product spaces

- You have studied vector spaces. Now we introduce a new structure on them called 'inner product'.
- This abstractly captures the notions of 'length' and 'angle'.
- Once we have an inner product we can talk about the distance between elements of the vector space. This allows us to define convergence of a sequence vectors.

Lecture 7: Inner product spaces

- You have studied vector spaces. Now we introduce a new structure on them called 'inner product'.
- This abstractly captures the notions of 'length' and 'angle'.
- Once we have an inner product we can talk about the distance between elements of the vector space. This allows us to define convergence of a sequence vectors.
- The notion of inner product also allows us to define as to when one vector is 'orthogonal'to another.

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.
- We recall that for any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $\bar{z} := a - ib$.

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.
- We recall that for any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $\overline{z} := a - ib$.
- For any two complex numbers z, w we have $\overline{(zw)} = \overline{z}.\overline{w}$.

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.
- We recall that for any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $\overline{z} := a - ib$.
- For any two complex numbers z, w we have $\overline{(zw)} = \overline{z}.\overline{w}$.
- For $z \in \mathbb{C}$ as above,

$$|z| := (z\overline{z})^{\frac{1}{2}} = (a^2 + b^2)^{\frac{1}{2}}.$$

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.
- We recall that for any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $\bar{z} := a - ib$.
- For any two complex numbers z, w we have $\overline{(zw)} = \bar{z}.\bar{w}$.
- For $z \in \mathbb{C}$ as above,

$$|z| := (z\bar{z})^{\frac{1}{2}} = (a^2 + b^2)^{\frac{1}{2}}.$$

- Recall that any complex number $z \neq 0$ has the unique polar decomposition as $z = re^{i\theta}$ where $r = |z|$ and $0 \leq \theta < 2\pi$.

The setting

- In the following the field $\mathbb{F}$ would be either $\mathbb{R}$ or $\mathbb{C}$.
- We recall that for any complex number $z = a + ib$ with $a, b \in \mathbb{R}$, $\bar{z} := a - ib$.
- For any two complex numbers z, w we have $\overline{(zw)} = \bar{z}.\bar{w}$.
- For $z \in \mathbb{C}$ as above,

$$|z| := (z\bar{z})^{\frac{1}{2}} = (a^2 + b^2)^{\frac{1}{2}}.$$

- Recall that any complex number $z \neq 0$ has the unique polar decomposition as $z = re^{i\theta}$ where $r = |z|$ and $0 \leq \theta < 2\pi$.
- We have $|z| = 0$ if and only if $z = 0$. Further, $|zw| = |z||w|$ and $|z + w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$.

Standard inner products

- Consider a vector space V over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$.

Standard inner products

- Consider a vector space V over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product between two vectors x, y in V, usually denoted by $\langle x, y \rangle$ would be a scalar (an element of $\mathbb{F}$).

Standard inner products

- Consider a vector space V over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product between two vectors x, y in V, usually denoted by $\langle x, y \rangle$ would be a scalar (an element of $\mathbb{F}$).
- The main examples we have in mind are the following:

Standard inner products

- Consider a vector space V over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product between two vectors x, y in V, usually denoted by $\langle x, y \rangle$ would be a scalar (an element of $\mathbb{F}$).
- The main examples we have in mind are the following:
- Definition 7.1: For $n \in \mathbb{N}$, consider the vector space $\mathbb{R}^n$. The standard inner product on $\mathbb{R}^n$ is defined by:

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} x_j y_j .$$

Standard inner products

- Consider a vector space V over a field $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product between two vectors x, y in V, usually denoted by $\langle x, y \rangle$ would be a scalar (an element of $\mathbb{F}$).
- The main examples we have in mind are the following:
- Definition 7.1: For $n \in \mathbb{N}$, consider the vector space $\mathbb{R}^n$. The standard inner product on $\mathbb{R}^n$ is defined by:

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} x_j y_j .$$

- We note that $\langle x, x \rangle \geq 0$ for every x in $\mathbb{R}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

$\mathbb{C}^n$. as an inner product space

- Definition 7.2: For $n \in \mathbb{N}$, consider the vector space $\mathbb{C}^n$. The standard inner product on $\mathbb{C}^n$ is defined by:

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} \overline{x_j} y_j .$$

$\mathbb{C}^n$. as an inner product space

- Definition 7.2: For $n \in \mathbb{N}$, consider the vector space $\mathbb{C}^n$. The standard inner product on $\mathbb{C}^n$ is defined by:

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} \overline{x_j} y_j .$$

- Here also we note that $\langle x, x \rangle \geq 0$ for every x in $\mathbb{C}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

such that

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

  such that
- (i) $\langle x, cy + dz \rangle = c\langle x, y \rangle + d\langle x, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in second variable.)

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

such that

- (i) $\langle x, cy + dz \rangle = c\langle x, y \rangle + d\langle x, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in second variable.)
- (ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$. (Conjugate Symmetry.)

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

  such that
- (i) $\langle x, cy + dz \rangle = c\langle x, y \rangle + d\langle x, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in second variable.)
- (ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$. (Conjugate Symmetry.)
- (iii) $\langle x, x \rangle \geq 0$ for all x in V. (Positivity.)

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

  such that
- (i) $\langle x, cy + dz \rangle = c\langle x, y \rangle + d\langle x, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in second variable.)
- (ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$. (Conjugate Symmetry.)
- (iii) $\langle x, x \rangle \geq 0$ for all x in V. (Positivity.)
- (iv) $\langle x, x \rangle = 0$ if and only if $x = 0$. (Definiteness.)

Definitions

- Definition 7.3: Let V be a vector space over a field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.
- An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

  such that
- (i) $\langle x, cy + dz \rangle = c\langle x, y \rangle + d\langle x, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in second variable.)
- (ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$. (Conjugate Symmetry.)
- (iii) $\langle x, x \rangle \geq 0$ for all x in V. (Positivity.)
- (iv) $\langle x, x \rangle = 0$ if and only if $x = 0$. (Definiteness.)
- Some authors take inner product as linear in the first variable. It is a matter of convention. A vector space with a specified inner product is called an inner product space.

Examples

- It is clear that standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner products.

Examples

- It is clear that standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner products.
- Example 7.4: For $n \in \mathbb{N}$, fix scalars $a_1, a_2, \ldots, a_n$.

Examples

- It is clear that standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner products.
- Example 7.4: For $n \in \mathbb{N}$, fix scalars $a_1, a_2, \ldots, a_n$.
- Then

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} a_j x_j y_j ,$$

Examples

- It is clear that standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner products.
- Example 7.4: For $n \in \mathbb{N}$, fix scalars $a_1, a_2, \ldots, a_n$.
- Then

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} a_j x_j y_j,$$

- is an inner-product on $\mathbb{R}^n$ if and only if $a_j > 0$ for every j.

Examples

- It is clear that standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner products.
- Example 7.4: For $n \in \mathbb{N}$, fix scalars $a_1, a_2, \ldots, a_n$.
- Then

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} a_j x_j y_j,$$

- is an inner-product on $\mathbb{R}^n$ if and only if $a_j > 0$ for every j.
- Note that if $a_j \geq 0$, then conditions (i)-(iii) of the inner product are satisfied but the definiteness may not be satisfied. In such cases, $\langle \cdot, \cdot \rangle$ is known as semi-inner product.

Linearity or anti-linearity in the first variable

- If the field $\mathbb{F} = \mathbb{R}$, from linearity and (conjugate) symmetry of the inner-product we get also linearity in the first variable.

Linearity or anti-linearity in the first variable

- If the field $\mathbb{F} = \mathbb{R}$, from linearity and (conjugate) symmetry of the inner-product we get also linearity in the first variable.
- That is, $\langle cx + dy, z \rangle = c\langle x, z \rangle + d\langle y, z \rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in first variable in the real case.)

Linearity or anti-linearity in the first variable

- If the field $\mathbb{F} = \mathbb{R}$, from linearity and (conjugate) symmetry of the inner-product we get also linearity in the first variable.
- That is, $\langle cx + dy, z\rangle = c\langle x, z\rangle + d\langle y, z\rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in first variable in the real case.)
- On the other hand if the field $\mathbb{F} = \mathbb{C}$, from linearity and conjugate symmetry of the inner-product we get anti-linearity in the first variable.

Linearity or anti-linearity in the first variable

- If the field $\mathbb{F} = \mathbb{R}$, from linearity and (conjugate) symmetry of the inner-product we get also linearity in the first variable.
- That is, $\langle cx + dy, z\rangle = c\langle x, z\rangle + d\langle y, z\rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$ (Linearity in first variable in the real case.)
- On the other hand if the field $\mathbb{F} = \mathbb{C}$, from linearity and conjugate symmetry of the inner-product we get anti-linearity in the first variable.
- That is, $\langle cx + dy, z\rangle = \bar{c}\langle x, z\rangle + \bar{d}\langle y, z\rangle$, for all $x, y, z \in V, c, d \in \mathbb{F}$.

Another example

- Take $\mathbb{F} = \mathbb{C}$. Fix $m, n \in \mathbb{N}$. Then we know that $M_{m,n}(\mathbb{C})$ is a vector space of dimension mn over $\mathbb{C}$.

Another example

- Take $\mathbb{F} = \mathbb{C}$. Fix $m, n \in \mathbb{N}$. Then we know that $M_{m,n}(\mathbb{C})$ is a vector space of dimension mn over $\mathbb{C}$.
- For $X, Y \in M_{m,n}(\mathbb{C})$ take

$$\langle X, Y \rangle = \quad \text{trace}(X^* Y)$$

where $(X^*)_{jk} = \overline{x_{kj}}, 1 \leq k \leq m; 1 \leq j \leq n$.

Another example

- Take $\mathbb{F} = \mathbb{C}$. Fix $m, n \in \mathbb{N}$. Then we know that $M_{m,n}(\mathbb{C})$ is a vector space of dimension mn over $\mathbb{C}$.
- For $X, Y \in M_{m,n}(\mathbb{C})$ take

$$\langle X, Y \rangle = \quad \text{trace}(X^* Y)$$

  where $(X^*)_{jk} = \overline{x_{kj}}, 1 \leq k \leq m; 1 \leq j \leq n$.
- Then $\langle \cdot, \cdot \rangle$ is an inner product on $M_{m,n}(\mathbb{C})$.

Another example

- Take $\mathbb{F} = \mathbb{C}$. Fix $m, n \in \mathbb{N}$. Then we know that $M_{m,n}(\mathbb{C})$ is a vector space of dimension mn over $\mathbb{C}$.
- For $X, Y \in M_{m,n}(\mathbb{C})$ take

$$\langle X, Y \rangle = \operatorname{trace}(X^* Y)$$

where $(X^*)_{jk} = \overline{x_{kj}}, 1 \leq k \leq m; 1 \leq j \leq n$.
- Then $\langle \cdot, \cdot \rangle$ is an inner product on $M_{m,n}(\mathbb{C})$.
- Proof: We have,

$$\begin{aligned} \langle X, Y \rangle &= \operatorname{trace}(X^* Y) \\ &= \sum_{j=1}^{n} (X^* Y)_{jj} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} (X^*)_{jk} (Y)_{kj} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{m} \overline{x_{kj}} y_{kj} \end{aligned}$$

Continuation

- Now it is clear that this is essentially the standard inner product on $\mathbb{C}^{mn}$.

Continuation

- Now it is clear that this is essentially the standard inner product on $\mathbb{C}^{mn}$.
- Similarly,
$$\langle X, Y \rangle = \text{trace}(X^t Y),$$
is an inner product on $M_{m,n}(\mathbb{R})$.

$\mathbb{R}^2$.

- On $\mathbb{R}^2$, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$(\langle x, x \rangle)^{\frac{1}{2}} = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

is the distance of point x from the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\mathbb{R}^2$.

- On $\mathbb{R}^2$, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$(\langle x, x \rangle)^{\frac{1}{2}} = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

is the distance of point x from the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- More generally the distance between any two points $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is

$$((x_1 - y_1)^2 + (x_2 - y_2)^2))^{\frac{1}{2}} = (\langle x - y, x - y \rangle)^{\frac{1}{2}}.$$

$\mathbb{R}^2$.

- On $\mathbb{R}^2$, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$$(\langle x, x \rangle)^{\frac{1}{2}} = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

  is the distance of point x from the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

- More generally the distance between any two points $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is

$$((x_1 - y_1)^2 + (x_2 - y_2)^2))^{\frac{1}{2}} = (\langle x - y, x - y \rangle)^{\frac{1}{2}}.$$

- This suggests the following definitions.

The norm on an inner product space

- Definition 7.5: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the norm of a vector $x \in V$ is defined as

$$\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}.$$

The norm on an inner product space

- Definition 7.5: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the norm of a vector $x \in V$ is defined as

$$\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}.$$

- If x, y are vectors in V, the distance of y from x is defined as

$$d(x, y) = \|y - x\|.$$

The norm on an inner product space

- Definition 7.5: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the norm of a vector $x \in V$ is defined as

$$\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}.$$

- If x, y are vectors in V, the distance of y from x is defined as

$$d(x, y) = \|y - x\|.$$

- The 'distance function' $d : V \times V \to \mathbb{R}$ is also known as metric.

Basic properties of the norm

- Theorem 7.6: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\| \cdot \|$ be the associated norm on V. Then

Basic properties of the norm

- Theorem 7.6: Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the associated norm on V. Then
- (i) $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0$ if and only if $x = 0$.

Basic properties of the norm

- Theorem 7.6: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\| \cdot \|$ be the associated norm on V. Then
- (i) $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0$ if and only if $x = 0$.
- (ii) $\|ax\| = |a|\|x\|, \ \ \forall a \in \mathbb{F}$ and $x \in V$.

Basic properties of the norm

- Theorem 7.6: Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\|$ be the associated norm on V. Then
- (i) $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0$ if and only if $x = 0$.
- (ii) $\|ax\| = |a|\|x\|, \quad \forall a \in \mathbb{F}$ and $x \in V$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in V$.

Continuation

- Proof. (i) This is clear, as $\langle x, x\rangle \geq 0$ for all x and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Continuation

- Proof. (i) This is clear, as $\langle x, x\rangle \geq 0$ for all x and $\langle x, x\rangle = 0$ if and only if $x = 0$.
- (ii) We have,

$$\langle ax, ax\rangle = \bar{a}.a\langle x, x\rangle = |a|^2\langle x, x\rangle$$

from anti-linearity of the inner product in the first variable and linear in the second variable. Now (ii) is immediate.

Continuation

- Proof. (i) This is clear, as $\langle x, x\rangle \geq 0$ for all x and $\langle x, x\rangle = 0$ if and only if $x = 0$.
- (ii) We have,

$$\langle ax, ax\rangle = \bar{a}.a\langle x, x\rangle = |a|^2\langle x, x\rangle$$

  from anti-linearity of the inner product in the first variable and linear in the second variable. Now (ii) is immediate.
- (iii) This is a consequence of Cauchy-Schwarz inequality and we will prove it in the next class.

Continuation

- Proof. (i) This is clear, as $\langle x, x\rangle \geq 0$ for all x and $\langle x, x\rangle = 0$ if and only if $x = 0$.
- (ii) We have,

$$\langle ax, ax\rangle = \bar{a}.a\langle x, x\rangle = |a|^2\langle x, x\rangle$$

  from anti-linearity of the inner product in the first variable and linear in the second variable. Now (ii) is immediate.
- (iii) This is a consequence of Cauchy-Schwarz inequality and we will prove it in the next class.
- END OF LECTURE 7.