LINEAR ALGEBRA -II

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Lecture 8: Cauchy-Schwarz inequality

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- We note that $\langle x, x \rangle \geq 0$ for every x in $\mathbb{R}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

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- Here also we note that $\langle x, x \rangle \geq 0$ for every x in $\mathbb{C}^n$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

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- Some authors take inner product as linear in the first variable. It is a matter of convention. A vector space with a specified inner product is called an inner product space.

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- On $\mathbb{R}^2$, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

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- This suggests the following definitions.

The norm on an inner product space

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- The 'distance function' $d : V \to V \to \mathbb{R}$ is also known as metric.

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- (ii) $\|ax\| = |a|\|x\|, \quad \forall a \in \mathbb{F}$ and $x \in V$.

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- (iii) $\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in V$.

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from anti-linearity of the inner product in the first variable and linear in the second variable. Now (ii) is immediate.

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  from anti-linearity of the inner product in the first variable and linear in the second variable. Now (ii) is immediate.
- (iii) This is a consequence of Cauchy-Schwarz inequality and will be proved in the next class.

Cauchy-Schwarz inequality

- You may have seen the inequality
$$\sum_{j=1}^{n} a_j b_j \leq (\sum_{j=1}^{n} a_j^2)^{\frac{1}{2}} (\sum_{j=1}^{n} b_j^2)^{\frac{1}{2}}$$
for $n \in \mathbb{N}$ and real numbers $a_j, b_j, 1 \leq j \leq n$. Here we have a generalization of this result. Cauchy-Schwarz inequality is among the most important inequalities of mathematics.

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- Hence the required inequality is trivially true and is an equality.

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- Hence z is a scalar multiple of x. In particular, x, y are linearly dependent.
- Exercise: We proved the Cauchy-Schwarz inequality with the assumption of definiteness, that is $\langle x, x \rangle = 0$ implies $x = 0$. Prove the inequality without this assumption. (Hint: Consider the function $p(t) = \|y - t\langle x, y \rangle x\|^2$ for $t \in \mathbb{R}$.)

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- All these are immediate from respective properties of the norm.

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- Exercise: Find out as to why this is called parallelogram law.

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- Now the result is clear by taking the difference of two equations and dividing by four.

Continuation

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