LINEAR ALGEBRA -II

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Lecture 9: Gram-Schmidt orthogonalization

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- that is, given any vector $x \in V$, there exist, $c_1, c_2, \ldots, c_n$ in $\mathbb{F}$ such that $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$. Note that given x, these coefficients are uniquely determined due to linear independence of v_j's.

Orthogonality

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- as cross-terms are equal to zero due to orthogonality.

Basic properties

- Proposition 9.3: Suppose $\{v_1, v_2, \ldots, v_m\}$ is an orthogonal collection of non-zero vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then the collection $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

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- Proof. This is clear, as the dimension of V is same as the maximum possible size of linearly independent sets. ■

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- Definition 9.5: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then a basis $\{v_1, v_2, \ldots, v_n\}$ is said to be an orthonormal basis if

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } \ i \neq j. \end{cases}$$

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- Example 9.6: For $\mathbb{R}^n$ (or $\mathbb{C}^n$) the standard basis $\{e_1, e_2, \ldots, e_n\}$, where e_j is the vector whose j-th coordinate is one and all other coordinates are equal to zero, is an orthonormal basis with respect to the standard inner product.

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$$w = \sum_{j=1}^{n} \langle v_j, w \rangle v_j.$$

- Proof. As $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, $w = \sum_{i=1}^{n} c_i v_i$ for some $c_1, c_2, \ldots, c_n$ in $\mathbb{F}$.

Continuation

- Now for any j, using linearity of the inner product in second variable,

$$
\begin{aligned}
\langle v_j, w \rangle &= \langle v_j, \sum_{i=1}^{n} c_i v_i \rangle \\
&= \sum_{i=1}^{n} c_i \langle v_j, v_i \rangle \\
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- As this is true for every j, $w = \sum_{j=1}^{n} \langle v_j, w\rangle v_j$. ■

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- This way, $\{v_1, v_2\}$ are orthonormal (that is, they have norm one and are mutually orthogonal.)

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- We see that $\{v_1, \ldots, v_{k+1}\}$ are orthonormal and span $\{v_1, \ldots, v_{k+1}\} = \{u_1, \ldots, u_{k+1}\}$ so that the induction can be continued.

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- Continuing this way, up to $k = n$, we have an ortho-normal collection $\{v_1, v_2, \ldots, v_n\}$. Since it spans whole of V it is an orthonormal basis. ■

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- Exercise: Obtain an orthonormal basis for $\mathbb{R}^3$ by Gram-Schmidt orthogonalization applied to the basis:

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- END OF LECTURE 9.