LINEAR ALGEBRA -II

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Lecture 10: Adjoint of a linear map

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- More generally, two subsets S, T of V are said to be mutually orthogonal if

$$\langle u, v \rangle = 0, \quad \forall u \in S, v \in T.$$

Basic properties

- Proposition 9.3: Suppose $\{v_1, v_2, \ldots, v_m\}$ is an orthogonal collection of non-zero vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then the collection $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

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- For any j, $1 \leq j \leq m$, taking inner product with v_j, as $\langle v_j, v_i \rangle = \delta_{ij} \langle v_j, v_j \rangle$, we get

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- Proof. This is clear, as the dimension of V is same as the maximum possible size of linearly independent sets. ■

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- Definition 9.5: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then a basis $\{v_1, v_2, \ldots, v_n\}$ is said to be an orthonormal basis if

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- Example 9.6: For $\mathbb{R}^n$ (or $\mathbb{C}^n$) the standard basis $\{e_1, e_2, \ldots, e_n\}$, where e_j is the vector whose j-th coordinate is one and all other coordinates are equal to zero, is an orthonormal basis with respect to the standard inner product.

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- It gives a formula for the coefficients in the expansion of any vector in terms of the basis.
- Theorem 9.7: Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of an inner product space $(V, \langle\cdot,\cdot\rangle)$. Then for any vector $w \in V$,

$$w = \sum_{j=1}^{n} \langle v_j, w\rangle v_j.$$

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- Fix any $j, 1 \leq j \leq n$ and consider the basis vector v_j.

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- Fix any $j, 1 \leq j \leq n$ and consider the basis vector v_j.
- Now Tv_j is a vector in W and $\mathcal{C}$ is a basis for W.

Continuation

- Therefore, Tv_j is a linear combination of w_i's. Denote the corresponding coefficients as a_{ij}'s. That is, a_{ij} is determined by requiring:

$$Tv_j = \sum_{i=1}^{m} a_{ij} w_i, \quad 1 \leq j \leq n.$$

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- This defines the $m \times n$ matrix $A = [a_{ij}]_{1 \leq i \leq m; 1 \leq j \leq n}$ and is denoted as ${}_{\mathcal{C}}[T]_{\mathcal{B}}$. Observe that if $x = \sum_{j=1}^{n} x_j v_j$ then by linearity

$$\begin{aligned} Tx &= \sum_{j=1}^{n} x_j (Tv_j) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} x_j (a_{ij} w_i) \\ &= \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij} x_j] w_i. \end{aligned}$$

Continuation

- Conclusion: For a linear map $T : V \to W$, the matrix of T in bases $\mathcal{B}, C$ is the unique matrix A which satisfies

$$Tx = \sum_{i=1}^{m}[\sum_{j=1}^{n} a_{ij}x_j]w_i.$$

for $x = \sum_{j=1}^{n} x_j v_j$.

Maps on inner product spaces

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- Recall that for any vector $x \in V$, if $x = \sum_{j=1}^{n} x_j v_j$ then $x_j = \langle v_j, x \rangle$ so that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.

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- Similarly, considering the orthonormal basis $\mathcal{C}$ in W, for fixed j, $Tv_j = \sum_{i=1}^{m} a_{ij} w_i$ implies that $a_{ij} = \langle w_i, Tv_j \rangle$.

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- For general $x \in V$, we get

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- We summarize this as a theorem.

The matrix of a linear transformation under orthonormal bases

- Theorem 10.1: Let V, W be inner product spaces with orthonormal bases $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{C} = \{w_1, \ldots, w_m\}$ for some $m, n \in \mathbb{N}$. Let $T : V \to W$ be a linear map. Then the matrix of T in these bases is given by the $m \times n$ matrix $A = [a_{ij}]_{1 \leq i \leq m; 1 \leq j \leq n}$ where

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- Note that here:

$$Tv_j = \sum_{i=1}^{m} \langle w_i, Tv_j \rangle w_i = \sum_{i=1}^{m} a_{ij} w_i.$$

(Hermitian) adjoint

- Theorem 10.2: Let V, W be finite dimensional inner product spaces and let $T : V \to W$ be a linear map. Then there exists a unique linear map $S : W \to V$ satisfying

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- We know that A^* determines a linear map $S : W \to V$ satisfying

$$\langle v_j, Sw_i\rangle = (A^*)_{ji} = \overline{a_{ij}}.$$

Continuation

- Taking complex conjugation, we have, $\langle Sw_i, v_j \rangle = a_{ij}$ or

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is known as the (Hermitian) adjoint of T and is denoted by T^*.

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- Proof. Exercise.

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- Now from the uniqueness of the adjoint, we get $(TS)^* = S^*T^*$.

Composition

- Theorem 10.5: Let U, V, W be finite dimensional inner product spaces over a field $\mathbb{F}$. Let $S : U \to V$ and $T : V \to W$ be linear maps. Then

$$(TS)^* = S^* T^*.$$

- Proof. For $x \in U$ and $z \in W$,

$$\langle S^* T^* z, x \rangle = \langle T^* z, Sx \rangle = \langle z, TSx \rangle.$$

- Now from the uniqueness of the adjoint, we get $(TS)^* = S^* T^*$.
- END OF LECTURE 10.