LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 11: Isometries and unitaries

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$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } \ i \neq j. \end{cases}$$

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- In other words, an orthonormal basis is a basis consisting of mutually orthogonal unit vectors.
- Example 9.6: For $\mathbb{R}^n$ (or $\mathbb{C}^n$) the standard basis $\{e_1, e_2, \ldots, e_n\}$, where e_j is the vector whose j-th coordinate is one and all other coordinates are equal to zero, is an orthonormal basis with respect to the standard inner product.

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- It gives a formula for the coefficients in the expansion of any vector in terms of the basis.
- Theorem 9.7: Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then for any vector $w \in V$,

$$w = \sum_{j=1}^{n} \langle v_j, w \rangle v_j.$$

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- Let $T : V \to W$ be a linear map. We associate an $m \times n$ matrix A to T as described below and call it the matrix of T in bases $\mathcal{B}, C$
- Fix any $j, 1 \leq j \leq n$ and consider the basis vector v_j.
- Now Tv_j is a vector in W and $\mathcal{C}$ is a basis for W.

Continuation

- Therefore, Tv_j is a linear combination of w_i's. Denote the corresponding coefficients as a_{ij}'s. That is, a_{ij} is determined by requiring:

$$Tv_j = \sum_{i=1}^{m} a_{ij} w_i, \quad 1 \leq j \leq n.$$

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- This defines the $m \times n$ matrix $A = [a_{ij}]_{1 \leq i \leq m; 1 \leq j \leq n}$ and is denoted as ${}_{\mathcal{C}}[T]_{\mathcal{B}}$. Observe that if $x = \sum_{j=1}^{n} x_j v_j$ then by linearity

$$\begin{aligned} Tx &= \sum_{j=1}^{n} x_j (Tv_j) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} x_j (a_{ij} w_i) \\ &= \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij} x_j] w_i. \end{aligned}$$

Continuation

- Conclusion: For a linear map $T : V \to W$, the matrix of T in bases $\mathcal{B}, C$ is the unique matrix A which satisfies

$$Tx = \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij} x_j] w_i.$$

for $x = \sum_{j=1}^{n} x_j v_j$.

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- Similarly, considering the orthonormal basis $\mathcal{C}$ in W, for fixed j, $Tv_j = \sum_{i=1}^{m} a_{ij} w_i$ implies that $a_{ij} = \langle w_i, Tv_j \rangle$.

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- For general $x \in V$, we get

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- For general $x \in V$, we get

$$Tx = \sum_{i=1}^{m} [\sum_{j=1}^{n} \langle w_i, Tv_j\rangle \langle v_j, x\rangle] w_i$$

- We summarize this as a theorem.

The matrix of a linear transformation under orthonormal bases

- Theorem 10.1: Let V, W be inner product spaces with orthonormal bases $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{C} = \{w_1, \ldots, w_m\}$ for some $m, n \in \mathbb{N}$. Let $T : V \to W$ be a linear map. Then the matrix of T in these bases is given by the $m \times n$ matrix $A = [a_{ij}]_{1 \leq i \leq m; 1 \leq j \leq n}$ where

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- Conversely, given any $m \times n$ matrix $A = [a_{ij}]$, there exists unique linear map $T : V \to W$ satisfying

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- Note that here:

$$Tv_j = \sum_{i=1}^{m} \langle w_i, Tv_j \rangle w_i = \sum_{i=1}^{m} a_{ij} w_i.$$

(Hermitian) adjoint

- Theorem 10.2: Let V, W be finite dimensional inner product spaces and let $T : V \to W$ be a linear map. Then there exists a unique linear map $S : W \to V$ satisfying

$$\langle Sy, x\rangle = \langle y, Tx\rangle, \quad \forall x \in V, y \in W.$$

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- Proof. Choose an orthonormal basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V and an orthornormal basis $\mathcal{C} = \{w_1, \ldots, w_m\}$ for W. (Note that such orthonormal bases exist as we can apply Gram-Schmidt orthogonalization on some bases).

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- Consider the $n \times m$ matrix A^* defined by

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- We know that A^* determines a linear map $S : W \to V$ satisfying

$$\langle v_j, Sw_i\rangle = (A^*)_{ji} = \overline{a_{ij}}.$$

Continuation

- Taking complex conjugation, we have, $\langle Sw_i, v_j \rangle = a_{ij}$ or

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- Definition 10.3: Let V, W be finite dimensional inner product spaces and let $T : V \to W$ be a linear map. Then the unique linear map $S : W \to V$ satisfying

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is known as the (Hermitian) adjoint of T and is denoted by T^*.

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- Proof. Exercise.

Composition

- Theorem 10.5: Let U, V, W be finite dimensional inner product spaces over a field $\mathbb{F}$. Let $S : U \to V$ and $T : V \to W$ be linear maps. Then

$$(TS)^* = S^* T^*.$$

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- Proof. For $x \in U$ and $z \in W$,
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- Now from the uniqueness of the adjoint, we get $(TS)^* = S^*T^*$.

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- (ii) $S_2 : V \to W$ defined by

$$S_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

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Continuation

- (iii) $S_3 : V \to V$ defined by

$$S_3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{\sqrt{2}} \\ \frac{x_1-x_2}{\sqrt{2}} \end{pmatrix}$$

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- (ii) S preserves the metric:

$$d(Sx, Sy) = d(x, y), \quad \forall x, y \in V$$

where $d(x, y) = \|y - x\|, \quad x, y \in V$.

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- (iv) $S^*S = I_V$, where I_V denotes the identity of V.

Continuation

- Proof. $(i) \Rightarrow (ii)$. This is clear, as

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- $(i) \Rightarrow (iii)$. If $\mathbb{F} = \mathbb{R}$, we have

$$\begin{aligned}\langle Sx, Sy\rangle &= \frac{1}{4}(\langle S(x+y), S(x+y)\rangle - \langle S(x-y), S(x-y)\rangle) \\ &= \frac{1}{4}(\langle (x+y), (x+y)\rangle - \langle (x-y), (x-y)\rangle) \\ &= \langle x, y\rangle.\end{aligned}$$

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\langle Sx, Sy \rangle &= \frac{1}{4}(\langle S(x+y), S(x+y)\rangle - \langle S(x-y), S(x-y)\rangle) \\
&= \frac{1}{4}(\langle (x+y), (x+y)\rangle - \langle (x-y), (x-y)\rangle) \\
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- Similarly, using polarization identity we have the result for $\mathbb{F} = \mathbb{C}$.

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- Proof: Clear from (iii) of previous theorem. ■

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- The converse is an exercise.

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$$0 = \langle S(x-y), S(x-y) \rangle = \langle (x-y), (x-y) \rangle.$$

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- $(iii) \Rightarrow (i)$. Now as $S^*S = I_V$, S is isometric. As S is invertible, it is a bijection.

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- (v) Suppose $S : V \to V$ is an isometry then it is a unitary.
- Proof: Exercise.

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- END OF LECTURE 11.