LINEAR ALGEBRA -II

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Lecture 12: Orthogonal decomposition

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- Definition 9.5: Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Then a basis $\{v_1, v_2, \ldots, v_n\}$ is said to be an orthonormal basis if

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- In other words, an orthonormal basis is a basis consisting of mutually orthogonal unit vectors.
- Example 9.6: For $\mathbb{R}^n$ (or $\mathbb{C}^n$) the standard basis $\{e_1, e_2, \ldots, e_n\}$, where e_j is the vector whose j-th coordinate is one and all other coordinates are equal to zero, is an orthonormal basis with respect to the standard inner product.

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- It gives a formula for the coefficients in the expansion of any vector in terms of the basis.
- Theorem 9.7: Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then for any vector $w \in V$,

$$w = \sum_{j=1}^{n} \langle v_j, w \rangle v_j.$$

Orthogonal complement

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$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

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- Then

$$S^{\perp} = \{ \begin{pmatrix} c \\ c \\ c \end{pmatrix} : c \in \mathbb{R} \}.$$

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.

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- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw \rangle = c\langle x, v \rangle + d\langle x, w \rangle = c.0 + d.0 = 0.$$

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- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^\perp$ is a subspace of V. Further, $(S^\perp)^\perp$ is a subspace containing S.
- Proof: We recall the definition of $S^\perp$:

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- Hence $cv + dw \in S^\perp$. This proves that $S^\perp$ is a subspace of V.
- It is easy to see that if $x \in S$ then $x \in (S^\perp)^\perp$. Therefore $S \subseteq (S^\perp)^\perp$.
- We have already seen that orthogonal complement of any non-empty subset is a subspace. In particular, $(S^\perp)^\perp$ is a subspace.

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- Clearly,

$$V_2 = \{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \}.$$

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- We see that any vector $x \in V$ decomposes uniquely as $x = y + z$ with $y \in V_0$ and $z \in V_1$.

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- We want to show that this is a general phenomenon.

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.

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- If not, choose any $v_{k+1} \in V \backslash M_k$. Then $\{v_1, \dots, v_{k+1}\}$ is a linearly independent set (Why?). Take
$$M_{k+1} := \text{ span}\{v_1, \dots, v_{k+1}\}.$$

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$$M_{k+1} := \text{span}\{v_1, \ldots, v_{k+1}\}.$$

- If $V = M_{k+1}$ then $\{v_1, \ldots, v_{k+1}\}$ is a basis for V and we are done. If not, take $v_{k+2} \in V \backslash M_{k+1}$ and continue the induction process.

Continuation

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- The process terminates after a finite number of steps as V is finite dimensional and so it can have at most $\dim(V)$ linearly independent elements.
- Therefore $V = M_n$ for some n and $\{v_1, \ldots, v_n\}$ is a basis for V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

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- Proof: By the previous theorem we may extend $\{v_1, \dots, v_k\}$ to a basis $\{v_1, \dots, v_k, w_{k+1}, \dots, w_n\}$ of V.

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- Now apply the Gram-Schmidt procedure on $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ to get an ortho-normal basis $\{e_1, \ldots, e_n\}$ of V.

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- Now apply the Gram-Schmidt procedure on $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ to get an ortho-normal basis $\{e_1, \ldots, e_n\}$ of V.
- It is an elementary exercise to see that $e_j = v_j$ for $1 \leq j \leq k$ as $v_1, \ldots, v_k$ are already orthonormal. ■

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

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- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i \leq k$ and $(k+1) \leq j \leq n$
- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \ldots, c_n$.

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- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i \leq k$ and $(k+1) \leq j \leq n$
- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \dots, c_n$.
- This shows $\langle x, y \rangle = 0$ for all $x \in V_0$ and $y \in V_1$. Hence $V_1 \subseteq (V_0)^\perp$.

Continuation

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- As $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.

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- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.

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- Hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.

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- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.
- Hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.
- This proves $(V_0)^{\perp} \subseteq V_1$ and completes the proof of our claim.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as
$$x = y + z$$
where $y \in V_0$ and $z \in V_0^{\perp}$.

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- Proof: Suppose $V_0 = \{0\}$. Then $V_0^{\perp} = V$ and we can decompose x as $x = 0 + x$, with $0 \in V_0$ and $x \in V_0^{\perp}$.

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- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \ldots, v_n\}$ of V.

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- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \ldots, v_n\}$ of V.
- Now we know that any $x \in V$ decomposes as
$$x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$$

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
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- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.

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- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$

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$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$
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- Also as $z, z' \in V_0^{\perp}$, $y - y' = z' - z \in V_0^{\perp}$.
- Hence $\langle y - y', y - y' \rangle = 0$. Consequently $y = y'$ and $z' = z$. This proves the uniqueness.

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- Therefore any $x \in V$ decomposes as $x = \langle v, x \rangle v + z$ where z is orthogonal to v.

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- and the positivity of this is same as the Cauchy-Schwarz inequality:

$$\|x\|^2\|y\|^2 \geq \|\langle x, y\rangle|^2.$$

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- END OF LECTURE 12.