LINEAR ALGEBRA -II

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Lecture 14: Best approximation property of projections

- We recall: Definition 12.1. Let S be a non-empty subset of an inner product space V. Then the orthogonal complement of S is defined as:

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- Then

$$S^{\perp} = \{ \begin{pmatrix} c \\ c \\ c \end{pmatrix} : c \in \mathbb{R}\}.$$

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- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw \rangle = c\langle x, v \rangle + d\langle x, w \rangle = c.0 + d.0 = 0.$$

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- It is easy to see that if $x \in S$ then $x \in (S^\perp)^\perp$. Therefore $S \subseteq (S^\perp)^\perp$.
- We have already seen that orthogonal complement of any non-empty subset is a subspace. In particular, $(S^\perp)^\perp$ is a subspace.

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- Clearly,

$$V_1 = \{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \}.$$

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- We want to show that this is a general phenomenon.

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.

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- Proof: Take
$$M_k := \text{span}\,\{v_1, v_2, \dots, v_k\}$$
- If $M_k = V$ then $V_0 = V$, $\{v_1, \dots, v_k\}$ is a basis for V and so no extension is required.

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- If not, choose any $v_{k+1} \in V \setminus M_k$. Then $\{v_1, \ldots, v_{k+1}\}$ is a linearly independent set (Why?). Take

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$$M_{k+1} := \text{span}\{v_1, \ldots, v_{k+1}\}.$$

- If $V = M_{k+1}$ then $\{v_1, \ldots, v_{k+1}\}$ is a basis for V and we are done. If not, take $v_{k+2} \in V \backslash M_{k+1}$ and continue the induction process.

Continuation

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- The process terminates after a finite number of steps as V is finite dimensional and so it can have at most $\dim(V)$ linearly independent elements.
- Therefore $V = M_n$ for some n and $\{v_1, \ldots, v_n\}$ is a basis for V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

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- Proof: By the previous theorem we may extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ of V.
- Now apply the Gram-Schmidt procedure on $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ to get an ortho-normal basis $\{e_1, \ldots, e_n\}$ of V.

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- It is an elementary exercise to see that $e_j = v_j$ for $1 \leq j \leq k$ as $v_1, \ldots, v_k$ are already orthonormal. ■

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

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$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$
- We claim that $V_1 = (V_0)^{\perp}$ and $\{v_{k+1}, \ldots, v_n\}$ is an ortho-normal basis of V_1.

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- Therefore $\langle \sum_{i=1}^k c_i v_i, \sum_{j=(k+1)}^n c_j v_j \rangle$ for any scalars $c_1, \ldots, c_n$.

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- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \ldots, c_n$.
- This shows $\langle x, y \rangle = 0$ for all $x \in V_0$ and $y \in V_1$. Hence $V_1 \subseteq (V_0)^{\perp}$.

Continuation

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- As $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.

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- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.

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- Hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.

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- Hence $x = \sum_{j=k+1}^n \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.
- This proves $(V_0)^\perp \subseteq V_1$ and completes the proof of our claim.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as
$$x = y + z$$
where $y \in V_0$ and $z \in V_0^{\perp}$.

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- Proof: Suppose $V_0 = \{0\}$. Then $V_0^\perp = V$ and we can decompose x as $x = 0 + x$, with $0 \in V_0$ and $x \in V_0^\perp$.

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- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \ldots, v_n\}$ of V.

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- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \ldots, v_n\}$ of V.
- Now we know that any $x \in V$ decomposes as
$$x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$$

Uniqueness

- Take

$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$

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- We have,
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- We have,
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- Therefore $y - y' = z' - z$. As $y, y' \in V_0$, $y - y' \in V_0$.

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$$y + z = y' + z'.$$
- Therefore $y - y' = z' - z$. As $y, y' \in V_0$, $y - y' \in V_0$.
- Also as $z, z' \in V_0^{\perp}$, $y - y' = z' - z \in V_0^{\perp}$.
- Hence $\langle y - y', y - y'\rangle = 0$. Consequently $y = y'$ and $z' = z$. This proves the uniqueness.

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.

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- As shown in the previous lecture this is related to Cauchy-Schwarz inequality.

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  is a basis for V_0.
- Let us apply Gram-Schmidt on this to get an orthonormal basis for V_0.

Continuation

- We get the first vector as

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$$\begin{aligned} w_2 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \langle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rangle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}. \end{aligned}$$

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- Given $x \in \mathbb{R}^3$, it decomposes as $y + z$, where $y \in V_0$, $z \in V_0^{\perp}$.

$$\begin{aligned} y &= \langle v_1, x\rangle v_1 + \langle v_2, x\rangle v_2 \\ &= \frac{x_1 - x_2}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} + \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}} \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix} \end{aligned}$$

Continuation

- $z = \frac{1}{3}\begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$

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- It is easy to see that $y \in V_0$, $z \in (V_0)^{\perp}$ and $x = y + z$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
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- (v) $P_{V_1} = I - P$ where $V_1 = (V_0)^{\perp}$.

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- So assume $V_0 \neq \{0\}$.
- (i). Let $\{v_1, \dots, v_k\}$ be an orthonormal basis of V_0. Extend it to an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of V.
- Then we know that

$$P(x) = \sum_{j=1}^{k} \langle v_j, x \rangle v_j.$$

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- Since the inner product is linear in the second variable, P is a linear map. This proves (i).

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- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$. Therefore $Px = x$ implies
$$\sum_{j=k+1}^{n} \langle v_j, x \rangle v_j = 0.$$
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- Now if $Px = 0$, then $\sum_{j=1}^{k} \langle v_j, x\rangle v_j = 0$ and hence $x = \sum_{j=k+1}^{n} \langle v_j, x\rangle v_j$, that is, $x \in (V_0)^{\perp}$.

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- Conversely if $x \in (V_0)^{\perp}$, then $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$, and consequently $Px = 0$.

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- This proves (ii).

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- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).

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- Now $P(P(x)) = P(\sum_{j=1}^{k} c_j v_j) = \sum_{j=1}^{k} c_j v_j = Px$.
- Hence $P^2(x) = P(x)$ for every x, or $P^2 = P$.

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- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

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- This shows that $P^* = P$ from the defining property of the adjoint of P.

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- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

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- We have just revisited our formula for the expansion of x in terms of an orthonormal basis.

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

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- In general, $d(a, B)$ may not be attained at any point in B. Also when it is attained it may not be at some unique point in B.

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- In general, $d(a, B)$ may not be attained at any point in B. Also when it is attained it may not be at some unique point in B.
- Example 14.1: Take $V = \mathbb{R}^2$. Take $a = (1, 0)$. Consider $B_1 = \{(x_1, x_2) : x_1 < 0\}$ and $B_2 = \{(x_1, x_2) : |x_1 - 1| \geq 1\}$

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

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- Then $d(a, B_1) = 1$ is not attained at any point. $d(a, B_2) = 1$ gets attained at two points.

Best approximation property

- Theorem 14.2: Let V_0 be a subspace of an inner product space V. Let P be the projection onto V_0. Then for $x \in V$,

$$d(x, V_0) = d(x, Px).$$

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- Proof: Suppose $x = y + z$, is the unique decomposition of x, with $y \in V_0, z \in V_0^{\perp}$.

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- Then $P_1 x = \frac{1}{n}(x_1 + \cdots + x_n) =: \overline{x}$.

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- Exercise: Work out more examples.

Example

- Example 14.3: Consider $V = \mathbb{R}^2$. Let
  $V_0 = \{c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : c \in \mathbb{R}\}$ where θ is a fixed real number.
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- END OF LECTURE 14.