LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 15: Algebraic description of projections

- We recall: Definition 12.1. Let S be a non-empty subset of an inner product space V. Then the orthogonal complement of S is defined as:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

Lecture 15: Algebraic description of projections

- We recall: Definition 12.1. Let S be a non-empty subset of an inner product space V. Then the orthogonal complement of S is defined as:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

- Example 12.2: Consider $S \subset \mathbb{R}^3$ where

$$S = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$$

Lecture 15: Algebraic description of projections

- We recall: Definition 12.1. Let S be a non-empty subset of an inner product space V. Then the orthogonal complement of S is defined as:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

- Example 12.2: Consider $S \subset \mathbb{R}^3$ where

$$S = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$$

- Then

$$S^{\perp} = \{ \begin{pmatrix} c \\ c \\ c \end{pmatrix} : c \in \mathbb{R} \}.$$

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.
- Proof: We recall the definition of $S^{\perp}$:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.
- Proof: We recall the definition of $S^{\perp}$:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw \rangle = c\langle x, v \rangle + d\langle x, w \rangle = c.0 + d.0 = 0.$$

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.
- Proof: We recall the definition of $S^{\perp}$:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw \rangle = c\langle x, v \rangle + d\langle x, w \rangle = c.0 + d.0 = 0.$$

- Hence $cv + dw \in S^{\perp}$. This proves that $S^{\perp}$ is a subspace of V.

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.
- Proof: We recall the definition of $S^{\perp}$:

$$S^{\perp} = \{v \in V : \langle x, v \rangle = 0, \quad \forall x \in S\}.$$

- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw \rangle = c\langle x, v \rangle + d\langle x, w \rangle = c.0 + d.0 = 0.$$

- Hence $cv + dw \in S^{\perp}$. This proves that $S^{\perp}$ is a subspace of V.
- It is easy to see that if $x \in S$ then $x \in (S^{\perp})^{\perp}$. Therefore $S \subseteq (S^{\perp})^{\perp}$.

Continuation

- Proposition 12.2: Let S be a non-empty subset of an inner product space V. Then $S^{\perp}$ is a subspace of V. Further, $(S^{\perp})^{\perp}$ is a subspace containing S.
- Proof: We recall the definition of $S^{\perp}$:

$$S^{\perp} = \{v \in V : \langle x, v\rangle = 0, \quad \forall x \in S\}.$$

- Now if $v, w \in S^{\perp}$ and $c, d \in \mathbb{F}$: For $x \in S$,

$$\langle x, cv + dw\rangle = c\langle x, v\rangle + d\langle x, w\rangle = c.0 + d.0 = 0.$$

- Hence $cv + dw \in S^{\perp}$. This proves that $S^{\perp}$ is a subspace of V.
- It is easy to see that if $x \in S$ then $x \in (S^{\perp})^{\perp}$. Therefore $S \subseteq (S^{\perp})^{\perp}$.
- We have already seen that orthogonal complement of any non-empty subset is a subspace. In particular, $(S^{\perp})^{\perp}$ is a subspace.

$\mathbb{R}^2$ in $\mathbb{R}^3$

- Consider $V = \mathbb{R}^3$ with standard inner product.

$\mathbb{R}^2$ in $\mathbb{R}^3$

- Consider $V = \mathbb{R}^3$ with standard inner product.
- Consider the subspace

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \}$$

$\mathbb{R}^2$ in $\mathbb{R}^3$

- Consider $V = \mathbb{R}^3$ with standard inner product.
- Consider the subspace

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R}\}$$

- Take $V_1 = (V_0)^{\perp}$.

$\mathbb{R}^2$ in $\mathbb{R}^3$

- Consider $V = \mathbb{R}^3$ with standard inner product.
- Consider the subspace

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \}$$

- Take $V_1 = (V_0)^\perp$.
- Clearly,

$$V_1 = \{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} : x_3 \in \mathbb{R} \}.$$

Continuation

- We see that any vector $x \in V$ decomposes uniquely as $x = y + z$ with $y \in V_0$ and $z \in V_1$.

Continuation

- We see that any vector $x \in V$ decomposes uniquely as $x = y + z$ with $y \in V_0$ and $z \in V_1$.
- Indeed for
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
the only choice is:
$$y = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}; z = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}.$$

Continuation

- We see that any vector $x \in V$ decomposes uniquely as $x = y + z$ with $y \in V_0$ and $z \in V_1$.
- Indeed for
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
the only choice is:
$$y = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}; z = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}.$$
- We want to show that this is a general phenomenon.

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.

Extending bases and orthonormal bases

- **Theorem 12.4:** Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \dots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \dots, v_n\}$ such that $\{v_1, \dots, v_n\}$ is a basis of V.
- **Proof:** Take
$$M_k := \text{span}\,\{v_1, v_2, \dots, v_k\}$$

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.
- Proof: Take
$$M_k := \text{span}\,\{v_1, v_2, \ldots, v_k\}$$
- If $M_k = V$ then $V_0 = V$, $\{v_1, \ldots, v_k\}$ is a basis for V and so no extension is required.

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.
- Proof: Take
$$M_k := \text{span}\ \{v_1, v_2, \ldots, v_k\}$$
- If $M_k = V$ then $V_0 = V$, $\{v_1, \ldots, v_k\}$ is a basis for V and so no extension is required.
- If not, choose any $v_{k+1} \in V \backslash M_k$. Then $\{v_1, \ldots, v_{k+1}\}$ is a linearly independent set (Why?). Take
$$M_{k+1} := \text{span}\{v_1, \ldots, v_{k+1}\}.$$

Extending bases and orthonormal bases

- Theorem 12.4: Let V_0 be a non-trivial subspace of a finite dimensional vector space V. Then any basis of V_0 extends to a basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.
- Proof: Take
$$M_k := \text{span}\,\{v_1, v_2, \ldots, v_k\}$$
- If $M_k = V$ then $V_0 = V$, $\{v_1, \ldots, v_k\}$ is a basis for V and so no extension is required.
- If not, choose any $v_{k+1} \in V \backslash M_k$. Then $\{v_1, \ldots, v_{k+1}\}$ is a linearly independent set (Why?). Take
$$M_{k+1} := \text{ span}\{v_1, \ldots, v_{k+1}\}.$$
- If $V = M_{k+1}$ then $\{v_1, \ldots, v_{k+1}\}$ is a basis for V and we are done. If not, take $v_{k+2} \in V \backslash M_{k+1}$ and continue the induction process.

Continuation

- The process terminates after a finite number of steps as V is finite dimensional and so it can have at most $\dim(V)$ linearly independent elements.

Continuation

- The process terminates after a finite number of steps as V is finite dimensional and so it can have at most $\dim(V)$ linearly independent elements.
- Therefore $V = M_n$ for some n and $\{v_1, \ldots, v_n\}$ is a basis for V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \dots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \dots, v_n\}$ such that $\{v_1, \dots, v_n\}$ is an orthonormal basis of V.
- Proof: By the previous theorem we may extend $\{v_1, \dots, v_k\}$ to a basis $\{v_1, \dots, v_k, w_{k+1}, \dots, w_n\}$ of V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Proof: By the previous theorem we may extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ of V.
- Now apply the Gram-Schmidt procedure on $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ to get an ortho-normal basis $\{e_1, \ldots, e_n\}$ of V.

Extending orthonormal bases

- Theorem 12.5: Let V_0 be a non-trivial subspace of a finite dimensional inner product space V. Then any orthonormal basis of V_0 extends to an orthonormal basis of V, that is, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis of V_0 then there exists $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Proof: By the previous theorem we may extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ of V.
- Now apply the Gram-Schmidt procedure on $\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_n\}$ to get an ortho-normal basis $\{e_1, \ldots, e_n\}$ of V.
- It is an elementary exercise to see that $e_j = v_j$ for $1 \leq j \leq k$ as $v_1, \ldots, v_k$ are already orthonormal. ■

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$
- We claim that $V_1 = (V_0)^{\perp}$ and $\{v_{k+1}, \ldots, v_n\}$ is an ortho-normal basis of V_1.

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \dots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \dots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \dots, v_n\}.$$
- We claim that $V_1 = (V_0)^\perp$ and $\{v_{k+1}, \dots, v_n\}$ is an ortho-normal basis of V_1.
- The second part is obvious. We only need to prove $V_1 = (V_0)^\perp$.

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$
- We claim that $V_1 = (V_0)^\perp$ and $\{v_{k+1}, \ldots, v_n\}$ is an ortho-normal basis of V_1.
- The second part is obvious. We only need to prove $V_1 = (V_0)^\perp$.
- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \le i \le k$ and $(k+1) \le j \le n$

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \dots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \dots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \dots, v_n\}.$$
- We claim that $V_1 = (V_0)^{\perp}$ and $\{v_{k+1}, \dots, v_n\}$ is an ortho-normal basis of V_1.
- The second part is obvious. We only need to prove $V_1 = (V_0)^{\perp}$.
- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i \leq k$ and $(k+1) \leq j \leq n$
- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \dots, c_n$.

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$
- We claim that $V_1 = (V_0)^{\perp}$ and $\{v_{k+1}, \ldots, v_n\}$ is an ortho-normal basis of V_1.
- The second part is obvious. We only need to prove $V_1 = (V_0)^{\perp}$.
- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i \leq k$ and $(k+1) \leq j \leq n$
- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \ldots, c_n$.

Orthogonal complement of a subspace

- Consider the set up as above, that is, V_0 is a non-trivial subspace of a finite dimensional inner product space V. Suppose $\{v_1, \ldots, v_k\}$ is an orthonormal basis of V_0 and $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V.
- Take
$$V_1 = \text{ span } \{v_{k+1}, \ldots, v_n\}.$$
- We claim that $V_1 = (V_0)^\perp$ and $\{v_{k+1}, \ldots, v_n\}$ is an ortho-normal basis of V_1.
- The second part is obvious. We only need to prove $V_1 = (V_0)^\perp$.
- Note that $\langle v_i, v_j \rangle = 0$ for all $1 \leq i \leq k$ and $(k+1) \leq j \leq n$
- Therefore $\langle \sum_{i=1}^{k} c_i v_i, \sum_{j=(k+1)}^{n} c_j v_j \rangle$ for any scalars $c_1, \ldots, c_n$.
- This shows $\langle x, y \rangle = 0$ for all $x \in V_0$ and $y \in V_1$. Hence $V_1 \subseteq (V_0)^\perp$.

Continuation

- Suppose $x \in V_0^{\perp}$.

Continuation

- Suppose $x \in V_0^{\perp}$.
- As $\{v_1, \dots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.

Continuation

- Suppose $x \in V_0^\perp$.
- As $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.
- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.

Continuation

- Suppose $x \in V_0^\perp$.
- As $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.
- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.
- Hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.

Continuation

- Suppose $x \in V_0^\perp$.
- As $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V, we get $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$.
- As x is orthogonal to V_0, we get $\langle v_j, x \rangle = 0$ for $1 \leq j \leq k$.
- Hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$ and therefore $x \in V_1$.
- This proves $(V_0)^\perp \subseteq V_1$ and completes the proof of our claim.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as

$$x = y + z$$

where $y \in V_0$ and $z \in V_0^{\perp}$.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as
$$x = y + z$$
where $y \in V_0$ and $z \in V_0^{\perp}$.
- Proof: Suppose $V_0 = \{0\}$. Then $V_0^{\perp} = V$ and we can decompose x as $x = 0 + x$, with $0 \in V_0$ and $x \in V_0^{\perp}$.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as

$$x = y + z$$

  where $y \in V_0$ and $z \in V_0^{\perp}$.
- Proof: Suppose $V_0 = \{0\}$. Then $V_0^{\perp} = V$ and we can decompose x as $x = 0 + x$, with $0 \in V_0$ and $x \in V_0^{\perp}$.
- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \dots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \dots, v_n\}$ of V.

Projection theorem

- Theorem 12.6: Let V_0 be a subspace of a finite dimensional inner product space V. Then every $x \in V$ decomposes uniquely as
$$x = y + z$$
where $y \in V_0$ and $z \in V_0^{\perp}$.
- Proof: Suppose $V_0 = \{0\}$. Then $V_0^{\perp} = V$ and we can decompose x as $x = 0 + x$, with $0 \in V_0$ and $x \in V_0^{\perp}$.
- If $V_0 \neq \{0\}$, choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V_0. Extend it to an orthonormal basis $\{v_1, \ldots, v_n\}$ of V.
- Now we know that any $x \in V$ decomposes as
$$x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$$

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$
- Therefore $y - y' = z' - z$. As $y, y' \in V_0$, $y - y' \in V_0$.

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x\rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x\rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$
- Therefore $y - y' = z' - z$. As $y, y' \in V_0$, $y - y' \in V_0$.
- Also as $z, z' \in V_0^{\perp}$, $y - y' = z' - z \in V_0^{\perp}$.

Uniqueness

- Take
$$y = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$$
and
$$z = \sum_{j=(k+1)}^{n} \langle v_j, x \rangle v_j.$$
- Clearly $y \in V_0$ and $z \in V_0^{\perp}$. This proves the existence.
- Suppose $x = y + z$ and $x = y' + z'$ are two decompositions of x with $y, y' \in V_0$ and $z, z' \in V_0^{\perp}$.
- We have,
$$y + z = y' + z'.$$
- Therefore $y - y' = z' - z$. As $y, y' \in V_0$, $y - y' \in V_0$.
- Also as $z, z' \in V_0^{\perp}$, $y - y' = z' - z \in V_0^{\perp}$.
- Hence $\langle y - y', y - y' \rangle = 0$. Consequently $y = y'$ and $z' = z$. This proves the uniqueness.

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.
- Consider the one dimensional space $V_0 = \{cy : c \in \mathbb{F}\}$.

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.
- Consider the one dimensional space $V_0 = \{cy : c \in \mathbb{F}\}$.
- Now $\{v\}$ is an ortho-normal basis for V_0 where

$$v = \frac{y}{\|y\|}.$$

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.
- Consider the one dimensional space $V_0 = \{cy : c \in \mathbb{F}\}$.
- Now $\{v\}$ is an ortho-normal basis for V_0 where

$$v = \frac{y}{\|y\|}.$$

- Therefore any $x \in V$ decomposes as $x = \langle v, x \rangle v + z$ where z is orthogonal to v.

A special case

- Suppose V is a finite dimensional inner product space and let y be a non-zero vector in V.
- Consider the one dimensional space $V_0 = \{cy : c \in \mathbb{F}\}$.
- Now $\{v\}$ is an ortho-normal basis for V_0 where

$$v = \frac{y}{\|y\|}.$$

- Therefore any $x \in V$ decomposes as $x = \langle v, x\rangle v + z$ where z is orthogonal to v.
- As shown in the previous lecture this is related to Cauchy-Schwarz inequality.

Example

- Example 13.1: Let $V = \mathbb{R}^n$ with the standard inner product. Let $V_0 = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}$.

Example

- Example 13.1: Let $V = \mathbb{R}^n$ with the standard inner product. Let $V_0 = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$.
- We first analyze the case when $n = 3$. Now $V = \mathbb{R}^3$ and

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 + x_3 = 0 \}.$$

Example

- Example 13.1: Let $V = \mathbb{R}^n$ with the standard inner product. Let $V_0 = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$.
- We first analyze the case when $n = 3$. Now $V = \mathbb{R}^3$ and

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 + x_3 = 0 \}.$$

- One can see that

$$\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \}$$

is a basis for V_0.

Example

- Example 13.1: Let $V = \mathbb{R}^n$ with the standard inner product. Let $V_0 = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$.
- We first analyze the case when $n = 3$. Now $V = \mathbb{R}^3$ and

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 + x_3 = 0 \}.$$

- One can see that

$$\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \}$$

  is a basis for V_0.
- Let us apply Gram-Schmidt on this to get an orthonormal basis for V_0.

Continuation

- We get the first vector as

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Continuation

- We get the first vector as

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}.$$

- Now take

$$\begin{aligned} w_2 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \left\langle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}. \end{aligned}$$

Continuation

- Now
$$v_2 = \frac{w_2}{\|w_2\|} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$$

Continuation

- Now
$$v_2 = \frac{w_2}{\|w_2\|} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$$
- $\{v_1, v_2\}$ is an ortho-normal basis for V_0.

Continuation

- Now
$$v_2 = \frac{w_2}{\|w_2\|} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$$
- $\{v_1, v_2\}$ is an ortho-normal basis for V_0.
- Given $x \in \mathbb{R}^3$, it decomposes as $y + z$, where $y \in V_0$, $z \in V_0^{\perp}$.

$$\begin{aligned} y &= \langle v_1, x\rangle v_1 + \langle v_2, x\rangle v_2 \\ &= \frac{x_1 - x_2}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} + \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}} \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix} \end{aligned}$$

Continuation

- $z = \frac{1}{3}\begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$

Continuation

- $z = \frac{1}{3}\begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$.
- For general n, with $\overline{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$,

$$y = \begin{pmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}, \quad z = \begin{pmatrix} \overline{x} \\ \overline{x} \\ \vdots \\ \overline{x} \end{pmatrix}$$

Continuation

- $z = \frac{1}{3}\begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$.
- For general n, with $\overline{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$,

$$y = \begin{pmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}, \quad z = \begin{pmatrix} \overline{x} \\ \overline{x} \\ \vdots \\ \overline{x} \end{pmatrix}$$

- It is easy to see that $y \in V_0$, $z \in (V_0)^{\perp}$ and $x = y + z$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.
- Theorem 13.3: Under the set up as above,

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.
- Theorem 13.3: Under the set up as above,
- (i) P is a linear map. (ii) $Px = x$ if and only if $x \in V_0$ and $Px = 0$ if and only if $x \in (V_0)^{\perp}$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.
- Theorem 13.3: Under the set up as above,
- (i) P is a linear map. (ii) $Px = x$ if and only if $x \in V_0$ and $Px = 0$ if and only if $x \in (V_0)^{\perp}$.
- (iii) $P(V) = V_0$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.
- Theorem 13.3: Under the set up as above,
- (i) P is a linear map. (ii) $Px = x$ if and only if $x \in V_0$ and $Px = 0$ if and only if $x \in (V_0)^{\perp}$.
- (iii) $P(V) = V_0$.
- (iv) $P = P^2 = P^*$.

Projection as a linear map

- Definition 13.2: Let V_0 be a subspace of a finite dimensional inner product space V. Then the projection on to V_0, is the map
$$P : V \to V$$
defined by
$$P(x) = y$$
where $x = y + z$, with $y \in V_0, z \in (V_0)^{\perp}$.
- Note that since every $x \in V$ decomposes uniquely as above, P is well-defined. If we want to emphasize the dependence of P on V_0, we may denote it by P_{V_0}.
- Theorem 13.3: Under the set up as above,
- (i) P is a linear map. (ii) $Px = x$ if and only if $x \in V_0$ and $Px = 0$ if and only if $x \in (V_0)^{\perp}$.
- (iii) $P(V) = V_0$.
- (iv) $P = P^2 = P^*$.
- (v) $P_{V_1} = I - P$ where $V_1 = (V_0)^{\perp}$.

Continuation

- Proof. If $V_0 = \{0\}$ then $P = 0$ and all the properties mentioned above are easy to see.

Continuation

- Proof. If $V_0 = \{0\}$ then $P = 0$ and all the properties mentioned above are easy to see.
- So assume $V_0 \neq \{0\}$.

Continuation

- Proof. If $V_0 = \{0\}$ then $P = 0$ and all the properties mentioned above are easy to see.
- So assume $V_0 \neq \{0\}$.
- (i). Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of V_0. Extend it to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of V.
- Then we know that

$$P(x) = \sum_{j=1}^{k} \langle v_j, x \rangle v_j.$$

(Note that P does not depend upon the choice of this basis!)

Continuation

- Proof. If $V_0 = \{0\}$ then $P = 0$ and all the properties mentioned above are easy to see.
- So assume $V_0 \neq \{0\}$.
- (i). Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of V_0. Extend it to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of V.
- Then we know that

$$P(x) = \sum_{j=1}^{k} \langle v_j, x \rangle v_j.$$

  (Note that P does not depend upon the choice of this basis!)
- Since the inner product is linear in the second variable, P is a linear map. This proves (i).

Continuation

- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$. Therefore $Px = x$ implies
$$\sum_{j=k+1}^{n} \langle v_j, x \rangle v_j = 0.$$
Therefore $x = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$ and hence $x \in V_0$.

Continuation

- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$. Therefore $Px = x$ implies

$$\sum_{j=k+1}^{n} \langle v_j, x \rangle v_j = 0.$$

  Therefore $x = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$ and hence $x \in V_0$.
- The converse is easy to see from the definition of P.

Continuation

- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$. Therefore $Px = x$ implies
$$\sum_{j=k+1}^{n} \langle v_j, x \rangle v_j = 0.$$
Therefore $x = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$ and hence $x \in V_0$.
- The converse is easy to see from the definition of P.
- Now if $Px = 0$, then $\sum_{j=1}^{k} \langle v_j, x \rangle v_j = 0$ and hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$, that is, $x \in (V_0)^{\perp}$.

Continuation

- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x\rangle v_j$. Therefore $Px = x$ implies
$$\sum_{j=k+1}^{n} \langle v_j, x\rangle v_j = 0.$$
Therefore $x = \sum_{j=1}^{k} \langle v_j, x\rangle v_j$ and hence $x \in V_0$.
- The converse is easy to see from the definition of P.
- Now if $Px = 0$, then $\sum_{j=1}^{k} \langle v_j, x\rangle v_j = 0$ and hence $x = \sum_{j=k+1}^{n} \langle v_j, x\rangle v_j$, that is, $x \in (V_0)^{\perp}$.
- Conversely if $x \in (V_0)^{\perp}$, then $x = \sum_{j=k+1}^{n} \langle v_j, x\rangle v_j$, and consequently $Px = 0$.

Continuation

- (ii). We know that $x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$. Therefore $Px = x$ implies
$$\sum_{j=k+1}^{n} \langle v_j, x \rangle v_j = 0.$$
Therefore $x = \sum_{j=1}^{k} \langle v_j, x \rangle v_j$ and hence $x \in V_0$.
- The converse is easy to see from the definition of P.
- Now if $Px = 0$, then $\sum_{j=1}^{k} \langle v_j, x \rangle v_j = 0$ and hence $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$, that is, $x \in (V_0)^{\perp}$.
- Conversely if $x \in (V_0)^{\perp}$, then $x = \sum_{j=k+1}^{n} \langle v_j, x \rangle v_j$, and consequently $Px = 0$.
- This proves (ii).

Continuation

- (iii). We want to show $P(V) = V_0$.

Continuation

- (iii). We want to show $P(V) = V_0$.
- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).

Continuation

- (iii). We want to show $P(V) = V_0$.
- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).

Continuation

- (iii). We want to show $P(V) = V_0$.
- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).
- (iv). If $x = \sum_{j=1}^{n} c_j v_j$, then $Px = \sum_{j=1}^{k} c_j v_j$.

Continuation

- (iii). We want to show $P(V) = V_0$.
- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).
- (iv). If $x = \sum_{j=1}^{n} c_j v_j$, then $Px = \sum_{j=1}^{k} c_j v_j$.
- Now $P(P(x)) = P(\sum_{j=1}^{k} c_j v_j) = \sum_{j=1}^{k} c_j v_j = Px$.

Continuation

- (iii). We want to show $P(V) = V_0$.
- From the formula given for P, $Px \in V_0$ for every $x \in V$ and hence $P(V) \subseteq V_0$. Since $Px = x$ for every $x \in V_0$, the range of P includes whole of V_0. This proves (iii).
- (iv). If $x = \sum_{j=1}^{n} c_j v_j$, then $Px = \sum_{j=1}^{k} c_j v_j$.
- Now $P(P(x)) = P(\sum_{j=1}^{k} c_j v_j) = \sum_{j=1}^{k} c_j v_j = Px$.
- Hence $P^2(x) = P(x)$ for every x, or $P^2 = P$.

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

- Note that $\langle y_i, z_j \rangle = 0$ for all i, j.

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

- Note that $\langle y_i, z_j \rangle = 0$ for all i, j.
- Now

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle.$$

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

- Note that $\langle y_i, z_j \rangle = 0$ for all i, j.
- Now

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle.$$

- Similarly,

$$\langle x_1, Px_2 \rangle = \langle y_1 + z_1, y_2 \rangle = \langle y_1, y_2 \rangle.$$

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

- Note that $\langle y_i, z_j \rangle = 0$ for all i, j.
- Now

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle.$$

- Similarly,

$$\langle x_1, Px_2 \rangle = \langle y_1 + z_1, y_2 \rangle = \langle y_1, y_2 \rangle.$$

- Consequently,

$$\langle Px_1, x_2 \rangle = \langle x_1, Px_2 \rangle$$

for all x_1, x_2 in V.

Continuation

- Suppose x_1, x_2 are in V. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ be the unique decompositions of x_1, x_2 so that

$$y_1, y_2 \in V_0; \quad z_1, z_2 \in V_0^{\perp}.$$

- Note that $\langle y_i, z_j \rangle = 0$ for all i, j.
- Now

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle.$$

- Similarly,

$$\langle x_1, Px_2 \rangle = \langle y_1 + z_1, y_2 \rangle = \langle y_1, y_2 \rangle.$$

- Consequently,

$$\langle Px_1, x_2 \rangle = \langle x_1, Px_2 \rangle$$

  for all x_1, x_2 in V.
- This shows that $P^* = P$ from the defining property of the adjoint of P.

Continuation

- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

$$P_{V_0}(x) = \sum_{j=1}^{k} c_j v_j, \quad P_{V_1}(x) = \sum_{j=k+1}^{n} c_j v_j.$$

Continuation

- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

$$P_{V_0}(x) = \sum_{j=1}^{k} c_j v_j, \quad P_{V_1}(x) = \sum_{j=k+1}^{n} c_j v_j.$$

- From these formulae, it is easy to see that $P_{V_1} = 1 - P_{V_0}$.

Continuation

- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

$$P_{V_0}(x) = \sum_{j=1}^{k} c_j v_j, \quad P_{V_1}(x) = \sum_{j=k+1}^{n} c_j v_j.$$

- From these formulae, it is easy to see that $P_{V_1} = 1 - P_{V_0}$.
- This completes the proof Theorem 13.2.

Continuation

- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

$$P_{V_0}(x) = \sum_{j=1}^{k} c_j v_j, \quad P_{V_1}(x) = \sum_{j=k+1}^{n} c_j v_j.$$

- From these formulae, it is easy to see that $P_{V_1} = 1 - P_{V_0}$.
- This completes the proof Theorem 13.2.
- Remark 13.4: Observe that $P_{\{0\}} = 0$ and $P_V = I$. In particular,

$$P_V(x) = x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$$

independent of the choice of the basis.

Continuation

- (v). If $x = \sum_{j=1}^{n} c_j v_j$,

$$P_{V_0}(x) = \sum_{j=1}^{k} c_j v_j, \quad P_{V_1}(x) = \sum_{j=k+1}^{n} c_j v_j.$$

- From these formulae, it is easy to see that $P_{V_1} = 1 - P_{V_0}$.
- This completes the proof Theorem 13.2.
- Remark 13.4: Observe that $P_{\{0\}} = 0$ and $P_V = I$. In particular,

$$P_V(x) = x = \sum_{j=1}^{n} \langle v_j, x \rangle v_j$$

independent of the choice of the basis.

- We have just revisited our formula for the expansion of x in terms of an orthonormal basis.

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

and

$$d(a, B) := \inf\{d(a, b) : b \in B\}.$$

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

and

$$d(a, B) := \inf\{d(a, b) : b \in B\}.$$

- We may informally call $d(A, B)$ as the distance between A, B and $d(a, B)$ as the distance between a and B. But note that now we may have $d(A, B) = 0$ without having $A = B$.

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then
$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$
and
$$d(a, B) := \inf\{d(a, b) : b \in B\}.$$
- We may informally call $d(A, B)$ as the distance between A, B and $d(a, B)$ as the distance between a and B. But note that now we may have $d(A, B) = 0$ without having $A = B$.
- In general, $d(a, B)$ may not be attained at any point in B. Also when it is attained it may not be at some unique point in B.

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

and

$$d(a, B) := \inf\{d(a, b) : b \in B\}.$$

- We may informally call $d(A, B)$ as the distance between A, B and $d(a, B)$ as the distance between a and B. But note that now we may have $d(A, B) = 0$ without having $A = B$.
- In general, $d(a, B)$ may not be attained at any point in B. Also when it is attained it may not be at some unique point in B.
- Example 14.1: Take $V = \mathbb{R}^2$. Take $a = (1, 0)$. Consider $B_1 = \{(x_1, x_2) : x_1 < 0\}$ and $B_2 = \{(x_1, x_2) : |x_1 - 1| \geq 1\}$

Distance between sets

- Notation: Let A, B be non-empty subsets of an inner product space V and let $a \in V$. Then

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$$

and

$$d(a, B) := \inf\{d(a, b) : b \in B\}.$$

- We may informally call $d(A, B)$ as the distance between A, B and $d(a, B)$ as the distance between a and B. But note that now we may have $d(A, B) = 0$ without having $A = B$.
- In general, $d(a, B)$ may not be attained at any point in B. Also when it is attained it may not be at some unique point in B.
- Example 14.1: Take $V = \mathbb{R}^2$. Take $a = (1, 0)$. Consider $B_1 = \{(x_1, x_2) : x_1 < 0\}$ and $B_2 = \{(x_1, x_2) : |x_1 - 1| \geq 1\}$
- Then $d(a, B_1) = 1$ is not attained at any point. $d(a, B_2) = 1$ gets attained at two points.

Best approximation property

- Theorem 14.2: Let V_0 be a subspace of an inner product space V. Let P be the projection onto V_0. Then for $x \in V$,

$$d(x, V_0) = d(x, Px).$$

Moreover, Px is the unique point v such that $d(x, v) = d(x, V_0)$.

Best approximation property

- Theorem 14.2: Let V_0 be a subspace of an inner product space V. Let P be the projection onto V_0. Then for $x \in V$,

$$d(x, V_0) = d(x, Px).$$

  Moreover, Px is the unique point v such that $d(x, v) = d(x, V_0)$.
- This theorem tells us that Px is the unique 'best approximation' for x in V_0.

Best approximation property

- Theorem 14.2: Let V_0 be a subspace of an inner product space V. Let P be the projection onto V_0. Then for $x \in V$,

$$d(x, V_0) = d(x, Px).$$

  Moreover, Px is the unique point v such that $d(x, v) = d(x, V_0)$.
- This theorem tells us that Px is the unique 'best approximation' for x in V_0.
- Proof: Suppose $x = y + z$, is the unique decomposition of x, with $y \in V_0, z \in V_0^{\perp}$.

Continuation

- We have $Px = y$. Now consider any $v \in V_0$. Due to orthogonality of $y - v$ and z, we get

Continuation

- We have $Px = y$. Now consider any $v \in V_0$. Due to orthogonality of $y - v$ and z, we get
- $$\begin{aligned} \|x - v\|^2 &= \|(y + z) - v\|^2 \\ &= \langle (y - v) + z, (y - v) + z \rangle \\ &= \|(y - v)\|^2 + \|z\|^2. \end{aligned}$$

Continuation

- We have $Px = y$. Now consider any $v \in V_0$. Due to orthogonality of $y - v$ and z, we get
-

$$
\begin{aligned}
\|x - v\|^2 &= \|(y + z) - v\|^2 \\
&= \langle (y - v) + z, (y - v) + z \rangle \\
&= \|(y - v)\|^2 + \|z\|^2.
\end{aligned}
$$

- Hence

$$\inf_{v \in V_0} \|x - v\|^2 = \|z\|^2$$

and the infimum is attained only at $v = y$.

Continuation

- We have $Px = y$. Now consider any $v \in V_0$. Due to orthogonality of $y - v$ and z, we get
- $$\begin{aligned} \|x - v\|^2 &= \|(y + z) - v\|^2 \\ &= \langle (y - v) + z, (y - v) + z \rangle \\ &= \|(y - v)\|^2 + \|z\|^2. \end{aligned}$$
- Hence
  $$\inf_{v \in V_0} \|x - v\|^2 = \|z\|^2$$
  and the infimum is attained only at $v = y$.
- This proves the theorem.

Continuation

- We have $Px = y$. Now consider any $v \in V_0$. Due to orthogonality of $y - v$ and z, we get
- $$\begin{aligned} \|x - v\|^2 &= \|(y + z) - v\|^2 \\ &= \langle (y - v) + z, (y - v) + z \rangle \\ &= \|(y - v)\|^2 + \|z\|^2. \end{aligned}$$
- Hence
  $$\inf_{v \in V_0} \|x - v\|^2 = \|z\|^2$$
  and the infimum is attained only at $v = y$.
- This proves the theorem.
- Note that we are using the 'Pythagoras theorem' of inner product spaces.

Example

- Consider the Example 13.1, where $V = \mathbb{R}^n$ and

Example

- Consider the Example 13.1, where $V = \mathbb{R}^n$ and
- $V_0 = \{x \in V : \sum_{j=1}^{n} x_j = 0\}$.

Example

- Consider the Example 13.1, where $V = \mathbb{R}^n$ and
- $V_0 = \{x \in V : \sum_{j=1}^n x_j = 0\}$.
- $$V_1 = (V_0)^\perp = \{ \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} : c \in \mathbb{R}\}.$$

Example

- Consider the Example 13.1, where $V = \mathbb{R}^n$ and
- $V_0 = \{x \in V : \sum_{j=1}^n x_j = 0\}$.
- $$V_1 = (V_0)^\perp = \{\begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} : c \in \mathbb{R}\}.$$
- Let P_1 be the projection onto V_1.

Example

- Consider the Example 13.1, where $V = \mathbb{R}^n$ and
- $V_0 = \{x \in V : \sum_{j=1}^n x_j = 0\}$.
- $$V_1 = (V_0)^\perp = \{ \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} : c \in \mathbb{R}\}.$$
- Let P_1 be the projection onto V_1.
- Then $P_1 x = \frac{1}{n}(x_1 + \cdots + x_n) =: \overline{x}$.

Continuation

- So the best approximation for $x = (x_1, \ldots, x_n)$ among constant sequences is $(\overline{x}, \ldots, \overline{x})$.

Continuation

- So the best approximation for $x = (x_1, \ldots, x_n)$ among constant sequences is $(\overline{x}, \ldots, \overline{x})$.
- In other words, we have proved the theorem

$$\inf_{c \in \mathbb{R}} \sum_{j=1}^{n} (x_j - c)^2 = \sum_{j=1}^{n} (x_j - \overline{x})^2.$$

Continuation

- So the best approximation for $x = (x_1, \ldots, x_n)$ among constant sequences is $(\overline{x}, \ldots, \overline{x})$.
- In other words, we have proved the theorem

$$\inf_{c \in \mathbb{R}} \sum_{j=1}^{n} (x_j - c)^2 = \sum_{j=1}^{n} (x_j - \overline{x})^2.$$

- This value is n times the variance of the tuple $\{x_1 \ldots, x_n\}$. In other words,

$$\inf_{c \in \mathbb{R}} \frac{1}{n} \sum_{j=1}^{n} (x_j - c)^2 = \text{ Var } \{x_1, \ldots, x_n\},$$

and the infimum gets attained only at $\overline{x}$.

Continuation

- So the best approximation for $x = (x_1, \ldots, x_n)$ among constant sequences is $(\overline{x}, \ldots, \overline{x})$.
- In other words, we have proved the theorem

$$\inf_{c \in \mathbb{R}} \sum_{j=1}^{n} (x_j - c)^2 = \sum_{j=1}^{n} (x_j - \overline{x})^2.$$

- This value is n times the variance of the tuple $\{x_1 \ldots, x_n\}$. In other words,

$$\inf_{c \in \mathbb{R}} \frac{1}{n} \sum_{j=1}^{n} (x_j - c)^2 = \text{ Var } \{x_1, \ldots, x_n\},$$

  and the infimum gets attained only at $\overline{x}$.
- Exercise: Work out more examples.

Example

- Example 14.3: Consider $V = \mathbb{R}^2$. Let
  $V_0 = \{c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : c \in \mathbb{R}\}$ where θ is a fixed real number.
  Write down the matrix of the projection onto V_0.

Example

- Example 14.3: Consider $V = \mathbb{R}^2$. Let $V_0 = \{c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : c \in \mathbb{R}\}$ where θ is a fixed real number. Write down the matrix of the projection onto V_0.
- Solution. We take

$$P = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} .$$

Example

- Example 14.3: Consider $V = \mathbb{R}^2$. Let $V_0 = \{c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : c \in \mathbb{R}\}$ where θ is a fixed real number. Write down the matrix of the projection onto V_0.
- Solution. We take

$$P = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} .$$

- Then for any vector $x \in \mathbb{R}^2$,

$$Px = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = v\langle v, x\rangle = \langle v, x\rangle . v,$$

Example

- Example 14.3: Consider $V = \mathbb{R}^2$. Let $V_0 = \{c \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : c \in \mathbb{R}\}$ where θ is a fixed real number. Write down the matrix of the projection onto V_0.
- Solution. We take

$$P = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} .$$

- Then for any vector $x \in \mathbb{R}^2$,

$$Px = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = v\langle v, x\rangle = \langle v, x\rangle . v,$$

- where

$$v = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} .$$

Continuation

- Observe that $\{v\}$ is an orthonormal basis for V_0.

Continuation

- Observe that $\{v\}$ is an orthonormal basis for V_0.
- Therefore P given as above, that is,

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix},$$

is the projection on to V_0.

Continuation

- Observe that $\{v\}$ is an orthonormal basis for V_0.
- Therefore P given as above, that is,

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix},$$

  is the projection on to V_0.
- You may verify $P = P^2 = P^*$ and $P(\mathbb{R}^2) = V_0$.

- Theorem 15.1: Let V be a finite dimensional inner product space. Then a linear map $P : V \to V$ is a projection if and only if
$$P = P^2 = P^*.$$

Algebraic description of projections

- Theorem 15.1: Let V be a finite dimensional inner product space. Then a linear map $P : V \to V$ is a projection if and only if
$$P = P^2 = P^*.$$
- Proof: We have already seen that if P is a projection then $P = P^2 = P^*$.

Algebraic description of projections

- Theorem 15.1: Let V be a finite dimensional inner product space. Then a linear map $P : V \to V$ is a projection if and only if
$$P = P^2 = P^*.$$
- Proof: We have already seen that if P is a projection then $P = P^2 = P^*$.
- Now suppose $P : V \to V$ is a linear map such that
$$P = P^2 = P^*.$$

Algebraic description of projections

- Theorem 15.1: Let V be a finite dimensional inner product space. Then a linear map $P : V \to V$ is a projection if and only if
$$P = P^2 = P^*.$$
- Proof: We have already seen that if P is a projection then $P = P^2 = P^*$.
- Now suppose $P : V \to V$ is a linear map such that
$$P = P^2 = P^*.$$
- Take $V_0 = P(V) = \{Px : x \in V\}$.

Algebraic description of projections

- Theorem 15.1: Let V be a finite dimensional inner product space. Then a linear map $P : V \to V$ is a projection if and only if
$$P = P^2 = P^*.$$
- Proof: We have already seen that if P is a projection then $P = P^2 = P^*$.
- Now suppose $P : V \to V$ is a linear map such that
$$P = P^2 = P^*.$$
- Take $V_0 = P(V) = \{Px : x \in V\}$.
- Clearly V_0 is a subspace of V.

Continuation

- We wish to show that P is the projection onto V_0.

Continuation

- We wish to show that P is the projection onto V_0.
- For $x \in V$, take $y = Px$ and $z = x - Px$.

Continuation

- We wish to show that P is the projection onto V_0.
- For $x \in V$, take $y = Px$ and $z = x - Px$.
- Clearly $x = y + z$ and $y \in V_0$.

Continuation

- We wish to show that P is the projection onto V_0.
- For $x \in V$, take $y = Px$ and $z = x - Px$.
- Clearly $x = y + z$ and $y \in V_0$.
- It suffices to show that $z \in V_0^{\perp}$.

Continuation

- Consider any $w \in V_0$. So $w = Pv$ for some $v \in V$.

Continuation

- Consider any $w \in V_0$. So $w = Pv$ for some $v \in V$.
- Now

$$
\begin{aligned}
\langle w, z \rangle &= \langle Pv, x - Px \rangle \\
&= \langle v, P^*(x - Px) \rangle \\
&= \langle v, P(x - Px) \rangle \\
&= \langle v, Px - P^2 x \rangle \\
&= \langle v, Px - Px \rangle \\
&= 0.
\end{aligned}
$$

Continuation

- Consider any $w \in V_0$. So $w = Pv$ for some $v \in V$.
- Now

$$
\begin{aligned}
\langle w, z\rangle &= \langle Pv, x - Px\rangle \\
&= \langle v, P^*(x - Px)\rangle \\
&= \langle v, P(x - Px)\rangle \\
&= \langle v, Px - P^2x\rangle \\
&= \langle v, Px - Px\rangle \\
&= 0.
\end{aligned}
$$

- This shows that $z \in V_0^\perp$. ■

Diagonalization of a projection

- Theorem 15.2: Let V_0 be a non-zero finite dimensional subspace of a finite dimensional inner product space V and let P the projection onto V_0. Then there exists an orthonormal basis $\mathcal{B}$ such that on $\mathcal{B}$, the matrix of P is given by

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is of order $k \times k$, with $k = \dim(V_0)$.

Diagonalization of a projection

- Theorem 15.2: Let V_0 be a non-zero finite dimensional subspace of a finite dimensional inner product space V and let P the projection onto V_0. Then there exists an orthonormal basis $\mathcal{B}$ such that on $\mathcal{B}$, the matrix of P is given by

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

  where I is of order $k \times k$, with $k = \dim(V_0)$.
- Proof: Let $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be an orthonormal basis for V, where $\{v_1, \ldots, v_k\}$ is an orthonormal basis for V_0.

Diagonalization of a projection

- Theorem 15.2: Let V_0 be a non-zero finite dimensional subspace of a finite dimensional inner product space V and let P the projection onto V_0. Then there exists an orthonormal basis $\mathcal{B}$ such that on $\mathcal{B}$, the matrix of P is given by

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

  where I is of order $k \times k$, with $k = \dim(V_0)$.
- Proof: Let $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be an orthonormal basis for V, where $\{v_1, \ldots, v_k\}$ is an orthonormal basis for V_0.
- Take $\mathcal{B} = \{v_1, \ldots, v_n\}$.

Diagonalization of a projection

- Theorem 15.2: Let V_0 be a non-zero finite dimensional subspace of a finite dimensional inner product space V and let P the projection onto V_0. Then there exists an orthonormal basis $\mathcal{B}$ such that on $\mathcal{B}$, the matrix of P is given by

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is of order $k \times k$, with $k = \dim(V_0)$.

- Proof: Let $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be an orthonormal basis for V, where $\{v_1, \ldots, v_k\}$ is an orthonormal basis for V_0.
- Take $\mathcal{B} = \{v_1, \ldots, v_n\}$.
- We have $Pv_j = v_j$ for $1 \leq j \leq k$ and $Pv_j = 0$ for $(k+1) \leq j \leq n$.

Eigenvalues of projections

- Theorem 15.3: Let V be a finite dimensional inner product space and let $P : V \to V$ be a projection. Suppose λ is an eigenvalue of P then $\lambda \in \{0, 1\}$.

Eigenvalues of projections

- Theorem 15.3: Let V be a finite dimensional inner product space and let $P : V \to V$ be a projection. Suppose λ is an eigenvalue of P then $\lambda \in \{0, 1\}$.
- Proof: This is clear as the characteristic polynomial of the matrix
$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$
where I is of order $k \times k$, is given by $p(x) = (x-1)^k x^{n-k}$. ■

Eigenvalues of projections

- Theorem 15.3: Let V be a finite dimensional inner product space and let $P : V \to V$ be a projection. Suppose λ is an eigenvalue of P then $\lambda \in \{0, 1\}$.
- Proof: This is clear as the characteristic polynomial of the matrix
$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$
where I is of order $k \times k$, is given by $p(x) = (x-1)^k x^{n-k}$. $\blacksquare$
- Note that 0 is the only eigenvalue of a projection P iff $P = 0$. Similarly 1 is the only eigenvalue of P if and only if $P = I$.

Continuation

- Consequently

$$\langle v_i, Pv_j \rangle = \begin{cases} \delta_{ij} & \text{if } 1 \leq i, j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Continuation

- Consequently

$$\langle v_i, Pv_j \rangle = \begin{cases} \delta_{ij} & \text{if } 1 \leq i, j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

- Hence the matrix of P on the orthonormal basis $\mathcal{B}$ is A as above.

Rank and trace

- Corollary 15.4: Let R be the matrix of a projection map on some orthonormal basis. Then trace $(R) = $ rank (R).

Rank and trace

- Corollary 15.4: Let R be the matrix of a projection map on some orthonormal basis. Then trace $(R) = \text{rank}\ (R)$.
- Proof: Clear from the description of the matrix of P.

Rank and trace

- Corollary 15.4: Let R be the matrix of a projection map on some orthonormal basis. Then trace $(R) =$ rank (R).
- Proof: Clear from the description of the matrix of P.
- Example 15.5: Show that $R : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \end{pmatrix}$$

is a projection. Write down the matrix of R on standard basis. Find a suitable orthonormal basis such that the entries of the matrix of R on that basis are 0's and 1's.

Rank and trace

- Corollary 15.4: Let R be the matrix of a projection map on some orthonormal basis. Then trace $(R) =$ rank (R).
- Proof: Clear from the description of the matrix of P.
- Example 15.5: Show that $R : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \end{pmatrix}$$

  is a projection. Write down the matrix of R on standard basis. Find a suitable orthonormal basis such that the entries of the matrix of R on that basis are 0's and 1's.

Rank and trace

- Corollary 15.4: Let R be the matrix of a projection map on some orthonormal basis. Then trace $(R) =$ rank (R).
- Proof: Clear from the description of the matrix of P.
- Example 15.5: Show that $R : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \\ \\ \frac{x_1+x_2+x_3}{3} \end{pmatrix}$$

  is a projection. Write down the matrix of R on standard basis. Find a suitable orthonormal basis such that the entries of the matrix of R on that basis are 0's and 1's.
- END OF LECTURE 15.