LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 16: Eigenvalues and eigenvectors

- Definition 16.1: Let A be a real/complex square matrix. Then the characteristic polynomial of A is defined as the polynomial

$$p(x) = \det(xI - A).$$

Lecture 16: Eigenvalues and eigenvectors

- Definition 16.1: Let A be a real/complex square matrix. Then the characteristic polynomial of A is defined as the polynomial

$$p(x) = \det(xI - A).$$

- If $A = [a_{ij}]_{1 \leq i,j \leq n}$

$$p(x) = \det \begin{bmatrix} x - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & x - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & x - a_{nn} \end{bmatrix}.$$

Lecture 16: Eigenvalues and eigenvectors

- Definition 16.1: Let A be a real/complex square matrix. Then the characteristic polynomial of A is defined as the polynomial

$$p(x) = \det(xI - A).$$

- If $A = [a_{ij}]_{1 \leq i,j \leq n}$

$$p(x) = \det \begin{bmatrix} x - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & x - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & x - a_{nn} \end{bmatrix}.$$

- Note that the characteristic polynomial of an $n \times n$ matrix is polynomial of degree n. Also its leading coefficient (the coefficient of x^n) is equal to 1. Such polynomials are known as monic polynomials.

Example

- Example 16.2: $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -3 & -8 \\ 0 & 0 & 2i \end{bmatrix}.$

Example

- Example 16.2: $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -3 & -8 \\ 0 & 0 & 2i \end{bmatrix}$.
- Then the characteristic polynomial of A is given by,

$$\begin{aligned}
p(x) &= \det(xI - A) \\
&= \det\left(\begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 0 & -3 & -8 \\ 0 & 0 & 2i \end{bmatrix}\right) \\
&= \det \begin{bmatrix} x-2 & +1 & 0 \\ 0 & x+3 & +8 \\ 0 & 0 & x-2i \end{bmatrix} \\
&= (x-2)(x+3)(x-2i) \\
&= (x^2+x-6)(x-2i) \\
&= x^3+x^2-6x-2ix^2-2ix+12i \\
&= x^3+(1-2i)x^2-(6+2i)x+12i
\end{aligned}$$

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

  for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.
- Note that $\lambda_1, \ldots, \lambda_n$ are roots of the polynomial, that is, these are the complex numbers λ such that $p(\lambda) = 0$.

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

  for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.
- Note that $\lambda_1, \ldots, \lambda_n$ are roots of the polynomial, that is, these are the complex numbers λ such that $p(\lambda) = 0$.
- Some of the roots could be repeated. The number of times a particular root λ appears in the factorization is known as the multiplicity of λ. A complex polynomial of degree n has exactly n roots (counting multiplicity).

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

  for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.
- Note that $\lambda_1, \ldots, \lambda_n$ are roots of the polynomial, that is, these are the complex numbers λ such that $p(\lambda) = 0$.
- Some of the roots could be repeated. The number of times a particular root λ appears in the factorization is known as the multiplicity of λ. A complex polynomial of degree n has exactly n roots (counting multiplicity).
- We will be assuming this theorem without giving proof.

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as
$$p(x) = a_n (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$
for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.
- Note that $\lambda_1, \ldots, \lambda_n$ are roots of the polynomial, that is, these are the complex numbers λ such that $p(\lambda) = 0$.
- Some of the roots could be repeated. The number of times a particular root λ appears in the factorization is known as the multiplicity of λ. A complex polynomial of degree n has exactly n roots (counting multiplicity).
- We will be assuming this theorem without giving proof.
- Example 16.4: Consider the polynomial $p(x) = x^2 + 1$.

Fundamental theorem of algebra

- Theorem 16.3(Fundamental theorem of algebra): Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial, with $n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{C}, a_n \neq 0$. Then p factorizes uniquely (up to permutation) as

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

  for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$.
- Note that $\lambda_1, \ldots, \lambda_n$ are roots of the polynomial, that is, these are the complex numbers λ such that $p(\lambda) = 0$.
- Some of the roots could be repeated. The number of times a particular root λ appears in the factorization is known as the multiplicity of λ. A complex polynomial of degree n has exactly n roots (counting multiplicity).
- We will be assuming this theorem without giving proof.
- Example 16.4: Consider the polynomial $p(x) = x^2 + 1$.
- We have $p(x) = (x + i)(x - i)$. So the roots of p can be complex even if the coefficients are real.

Eigenvalues and eigenvectors of complex matrices

- Definition 16.5: Let A be a complex square matrix. Then the eigenvalues of A are defined as roots of the characteristic polynomial of A. The algebraic multiplicity of an eigenvalue λ is defined as the multiplicity of λ in the factorization of the characteristic polynomial of A.

Eigenvalues and eigenvectors of complex matrices

- Definition 16.5: Let A be a complex square matrix. Then the eigenvalues of A are defined as roots of the characteristic polynomial of A. The algebraic multiplicity of an eigenvalue λ is defined as the multiplicity of λ in the factorization of the characteristic polynomial of A.
- An $n \times n$ matrix A is also viewed as the linear map $x \mapsto Ax$ on $\mathbb{C}^n$.

Eigenvalues and eigenvectors of complex matrices

- Definition 16.5: Let A be a complex square matrix. Then the eigenvalues of A are defined as roots of the characteristic polynomial of A. The algebraic multiplicity of an eigenvalue λ is defined as the multiplicity of λ in the factorization of the characteristic polynomial of A.
- An $n \times n$ matrix A is also viewed as the linear map $x \mapsto Ax$ on $\mathbb{C}^n$.
- Definition 16.6: Suppose A is an $n \times n$ complex matrix, and $\lambda \in \mathbb{C}$. If $x \in \mathbb{C}^n$ is a non-zero vector such that $Ax = \lambda x$, then x is said to be an eigenvector with eigenvalue λ.

Eigenvalues and eigenvectors of complex matrices

- Definition 16.5: Let A be a complex square matrix. Then the eigenvalues of A are defined as roots of the characteristic polynomial of A. The algebraic multiplicity of an eigenvalue λ is defined as the multiplicity of λ in the factorization of the characteristic polynomial of A.
- An $n \times n$ matrix A is also viewed as the linear map $x \mapsto Ax$ on $\mathbb{C}^n$.
- Definition 16.6: Suppose A is an $n \times n$ complex matrix, and $\lambda \in \mathbb{C}$. If $x \in \mathbb{C}^n$ is a non-zero vector such that $Ax = \lambda x$, then x is said to be an eigenvector with eigenvalue λ.
- It is to be noted that if $\lambda \in \mathbb{C}$ has an eigenvector x:

$$Ax = \lambda x.$$

  This means that $(\lambda I - A)x = 0$. In particular $(\lambda I - A)$ is not injective, therefore $\det(\lambda I - A) = 0$ or $p(\lambda) = 0$ where p is the characteristic polynomial of A. So λ is an eigenvalue.

Geometric multiplicity

- Definition 16.7: Let A be an $n \times n$ complex matrix. Then the geometric multiplicity of an eigenvalue λ is defined as the dimension of the kernel of $(\lambda I - A)$, that is, the dimension of the eigen space:

$$\{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

Geometric multiplicity

- Definition 16.7: Let A be an $n \times n$ complex matrix. Then the geometric multiplicity of an eigenvalue λ is defined as the dimension of the kernel of $(\lambda I - A)$, that is, the dimension of the eigen space:

$$\{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

- Example 16.8: Consider $B = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.

Geometric multiplicity

- Definition 16.7: Let A be an $n \times n$ complex matrix. Then the geometric multiplicity of an eigenvalue λ is defined as the dimension of the kernel of $(\lambda I - A)$, that is, the dimension of the eigen space:

$$\{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

- Example 16.8: Consider $B = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.
- Then the characteristic polynomial of B is given by $p(x) = (x-5)^3$. Therefore 5 appears with algebraic multiplicity 3.

Geometric multiplicity

- Definition 16.7: Let A be an $n \times n$ complex matrix. Then the geometric multiplicity of an eigenvalue λ is defined as the dimension of the kernel of $(\lambda I - A)$, that is, the dimension of the eigen space:

$$\{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

- Example 16.8: Consider $B = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$.
- Then the characteristic polynomial of B is given by $p(x) = (x-5)^3$. Therefore 5 appears with algebraic multiplicity 3.
- However, if x is an eigenvector with eigenvalue 5, we see

$$\begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 5 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Continuation

- That is,

$$
\begin{aligned}
5x_1 + x_2 &= 5x_1 \\
5x_2 + x_3 &= 5x_2 \\
5x_3 &= 5x_3
\end{aligned}
$$

Continuation

- That is,

$$\begin{aligned} 5x_1 + x_2 &= 5x_1 \\ 5x_2 + x_3 &= 5x_2 \\ 5x_3 &= 5x_3 \end{aligned}$$

- Solving this, we see

$$\{x : Bx = 5x\} = \{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \}$$

Therefore the geometric multiplicity of the eigenvalue 5 is 1.

Continuation

- More generally, for $n \geq 2$, and $c \in \mathbb{C}$, the $n \times n$ matrix

$$C = \begin{bmatrix} c & 1 & 0 & 0 & \dots \\ 0 & c & 1 & 0 & \dots \\ 0 & 0 & c & 1 & \dots \\ 0 & 0 & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Continuation

- More generally, for $n \geq 2$, and $c \in \mathbb{C}$, the $n \times n$ matrix

$$C = \begin{bmatrix} c & 1 & 0 & 0 & \dots \\ 0 & c & 1 & 0 & \dots \\ 0 & 0 & c & 1 & \dots \\ 0 & 0 & 0 & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

- has characteristic polynomial $(x - c)^n$. However, the geometric multiplicity of the eigenvalue c is just 1.

Comparing two multiplicities

- Theorem 16.9: Let A be an $n \times n$ complex matrix and let λ be an eigenvalue of A. Then

  $$1 \leq \text{geometric multiplicity of } \lambda \leq \text{ algebraic multiplicity of } \lambda.$$

Comparing two multiplicities

- Theorem 16.9: Let A be an $n \times n$ complex matrix and let λ be an eigenvalue of A. Then

  $$1 \leq \text{geometric multiplicity of } \lambda \leq \text{ algebraic multiplicity of } \lambda.$$

- Proof: We have already seen that whenever λ is an eigenvalue there exists non-zero x such that $Ax = \lambda x$. Hence the geometric multiplicity of λ is at least 1.

Comparing two multiplicities

- Theorem 16.9: Let A be an $n \times n$ complex matrix and let λ be an eigenvalue of A. Then

  $$1 \leq \text{geometric multiplicity of } \lambda \leq \text{ algebraic multiplicity of } \lambda.$$

- Proof: We have already seen that whenever λ is an eigenvalue there exists non-zero x such that $Ax = \lambda x$. Hence the geometric multiplicity of λ is at least 1.
- Now suppose the geometric multiplicity of λ is k. Then there exist k linearly independent vectors $\{w_1, w_2, \ldots, w_k\}$ such that $Aw_j = \lambda w_j$ for $1 \leq j \leq k$.

Continuation

- Extend $\{w_1, w_2, \ldots w_k\}$ to a basis $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of $\mathbb{C}^n$.

Continuation

- Extend $\{w_1, w_2, \ldots w_k\}$ to a basis $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of $\mathbb{C}^n$.
- Then the matrix of the linear map $x \mapsto Ax$ on this basis has the form:

$$B = \begin{bmatrix} \lambda I_k & C \\ 0 & D \end{bmatrix}$$

for some $C_{k \times n}, D_{(n-k)\times(n-k)}$, as $Aw_j = \lambda w_j$, for $1 \leq j \leq k$.

Continuation

- Extend $\{w_1, w_2, \ldots w_k\}$ to a basis $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of $\mathbb{C}^n$.
- Then the matrix of the linear map $x \mapsto Ax$ on this basis has the form:
$$B = \begin{bmatrix} \lambda I_k & C \\ 0 & D \end{bmatrix}$$
for some $C_{k \times n}, D_{(n-k)\times(n-k)}$, as $Aw_j = \lambda w_j$, for $1 \leq j \leq k$.
- Equivalently there exists an invertible matrix S such that $B = S^{-1}AS$.

Continuation

- Extend $\{w_1, w_2, \dots w_k\}$ to a basis $\{w_1, \dots, w_k, w_{k+1}, \dots, w_n\}$ of $\mathbb{C}^n$.
- Then the matrix of the linear map $x \mapsto Ax$ on this basis has the form:
$$B = \begin{bmatrix} \lambda I_k & C \\ 0 & D \end{bmatrix}$$
for some $C_{k\times n}, D_{(n-k)\times(n-k)}$, as $Aw_j = \lambda w_j$, for $1 \leq j \leq k$.
- Equivalently there exists an invertible matrix S such that $B = S^{-1}AS$.
- Observe that,

$$\begin{aligned} \det(xI - B) &= \det(xI - S^{-1}AS) \\ &= \det(xS^{-1}S - S^{-1}AS) \\ &= \det S^{-1}(xI - A)S \\ &= \det(S^{-1})\det(xI - A)\det(S) \\ &= \det(xI - A). \end{aligned}$$

Continuation

- Hence, the characteristic polynomial of A, has the form

$$\begin{aligned} p(x) &= \det(xI - \begin{bmatrix} \lambda I_k & C \\ 0 & D \end{bmatrix} \\ &= \det \begin{bmatrix} x - \lambda I_k & -C \\ 0 & xI - D \end{bmatrix} \\ &= (x-\lambda)^k . \det(xI - D). \end{aligned}$$

Continuation

- Hence, the characteristic polynomial of A, has the form

$$
\begin{aligned}
p(x) &= \det(xI - \begin{bmatrix} \lambda I_k & C \\ 0 & D \end{bmatrix} \\
&= \det \begin{bmatrix} x - \lambda I_k & -C \\ 0 & xI - D \end{bmatrix} \\
&= (x - \lambda)^k . \det(xI - D).
\end{aligned}
$$

- In particular, the algebraic multiplicity of λ is at least k. ■.

Real case

- Here also we will consider the eigenvalues of a real matrix as the (possibly complex) roots of the characteristic polynomial. The main issue here is that a polynomial with real coefficients can have complex roots.

Real case

- Here also we will consider the eigenvalues of a real matrix as the (possibly complex) roots of the characteristic polynomial. The main issue here is that a polynomial with real coefficients can have complex roots.
- For instance the characteristic polynomial of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is $p(x) = x^2 + 1 = (x - i)(x + i)$.

Real case

- Here also we will consider the eigenvalues of a real matrix as the (possibly complex) roots of the characteristic polynomial. The main issue here is that a polynomial with real coefficients can have complex roots.
- For instance the characteristic polynomial of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  is $p(x) = x^2 + 1 = (x - i)(x + i)$.
- Therefore the eigenvalues of a real matrix can be complex. The algebraic multiplicity would be the multiplicity in the associated characteristic polynomial.

Real case

- Here also we will consider the eigenvalues of a real matrix as the (possibly complex) roots of the characteristic polynomial. The main issue here is that a polynomial with real coefficients can have complex roots.
- For instance the characteristic polynomial of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  is $p(x) = x^2 + 1 = (x - i)(x + i)$.
- Therefore the eigenvalues of a real matrix can be complex. The algebraic multiplicity would be the multiplicity in the associated characteristic polynomial.
- However, we consider geometric multiplicity of an eigenvalue λ of a real matrix, considered as a linear map on $\mathbb{R}^n$, as the dimension of

$$\{x \in \mathbb{R}^n : Ax = \lambda x\}$$

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.
- (i) If $\lambda \notin \mathbb{R}$ then the geometric multiplicity of λ is zero.

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.
- (i) If $\lambda \notin \mathbb{R}$ then the geometric multiplicity of λ is zero.
- (ii) If $\lambda \in \mathbb{R}$ then the geometric multiplicity of λ is at least one.

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.
- (i) If $\lambda \notin \mathbb{R}$ then the geometric multiplicity of λ is zero.
- (ii) If $\lambda \in \mathbb{R}$ then the geometric multiplicity of λ is at least one.
- Proof: Now for $x \in \mathbb{R}^n$ clearly $Ax \in \mathbb{R}^n$. However as $\lambda \notin \mathbb{R}$ and $0 \neq x \in \mathbb{R}^n$, $\lambda x \notin \mathbb{R}^n$.

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.
- (i) If $\lambda \notin \mathbb{R}$ then the geometric multiplicity of λ is zero.
- (ii) If $\lambda \in \mathbb{R}$ then the geometric multiplicity of λ is at least one.
- Proof: Now for $x \in \mathbb{R}^n$ clearly $Ax \in \mathbb{R}^n$. However as $\lambda \notin \mathbb{R}$ and $0 \neq x \in \mathbb{R}^n$, $\lambda x \notin \mathbb{R}^n$.
- Therefore, $Ax = \lambda x$ is not possible.

Geometric multiplicity of real maps

- Theorem 16.10: Let A be a real matrix considered as a linear map on $\mathbb{R}^n$. Suppose λ is an eigenvalue of A.
- (i) If $\lambda \notin \mathbb{R}$ then the geometric multiplicity of λ is zero.
- (ii) If $\lambda \in \mathbb{R}$ then the geometric multiplicity of λ is at least one.
- Proof: Now for $x \in \mathbb{R}^n$ clearly $Ax \in \mathbb{R}^n$. However as $\lambda \notin \mathbb{R}$ and $0 \neq x \in \mathbb{R}^n$, $\lambda x \notin \mathbb{R}^n$.
- Therefore, $Ax = \lambda x$ is not possible.
- Hence the geometric multiplicity of non-real eigenvalues of real matrices (considered as real maps) is zero.

Continuation

- (ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue.

Continuation

- (ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue.
- Then $\det(\lambda I - A) = 0$. Hence $x \mapsto (\lambda I - A)x$ on $\mathbb{R}^n$ is not injective.

Continuation

- (ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue.
- Then $\det(\lambda I - A) = 0$. Hence $x \mapsto (\lambda I - A)x$ on $\mathbb{R}^n$ is not injective.
- In particular, there exists non-zero $x \in \mathbb{R}^n$ such that $Ax = \lambda x$.

Continuation

- (ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue.
- Then $\det(\lambda I - A) = 0$. Hence $x \mapsto (\lambda I - A)x$ on $\mathbb{R}^n$ is not injective.
- In particular, there exists non-zero $x \in \mathbb{R}^n$ such that $Ax = \lambda x$.
- Therefore, geometric multiplicity of any real eigenvalue of any real matrix is at least one.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.
- Now if λ is an eigenvalue of A, then $p(\lambda) = 0$.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.
- Now if λ is an eigenvalue of A, then $p(\lambda) = 0$.
- Now taking complex conjugate and observing that $\overline{a_j} = a_j$ for every j, we get $p(\overline{\lambda}) = 0$.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.
- Now if λ is an eigenvalue of A, then $p(\lambda) = 0$.
- Now taking complex conjugate and observing that $\overline{a_j} = a_j$ for every j, we get $p(\overline{\lambda}) = 0$.
- This shows that $(x - \lambda)(x - \bar{\lambda})$ is a factor of p.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.
- Now if λ is an eigenvalue of A, then $p(\lambda) = 0$.
- Now taking complex conjugate and observing that $\overline{a_j} = a_j$ for every j, we get $p(\bar{\lambda}) = 0$.
- This shows that $(x - \lambda)(x - \bar{\lambda})$ is a factor of p.
- We write $p(x) = (x - \lambda)(x - \bar{\lambda})q(x)$. Now $(x - (\bar{\lambda} + \lambda)x + |\lambda|^2)$ has only real coefficients. Then by the division algorithm, q also has only real coefficients.

Algebraic multiplicities of real matrices

- Theorem 16.11: Let A be a real square matrix. Then for any complex number λ the algebraic multiplicity of $\bar{\lambda}$ as an eigenvalue of A is same as the algebraic multiplicity of λ. (Eigenvalues appear in conjugate pairs.)
- Proof: Suppose

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  is the characteristic polynomial of A. Then $a_0, \ldots, a_{n-1}$ are real numbers.
- Now if λ is an eigenvalue of A, then $p(\lambda) = 0$.
- Now taking complex conjugate and observing that $\overline{a_j} = a_j$ for every j, we get $p(\overline{\lambda}) = 0$.
- This shows that $(x - \lambda)(x - \bar{\lambda})$ is a factor of p.
- We write $p(x) = (x - \lambda)(x - \bar{\lambda})q(x)$. Now $(x - (\overline{\lambda} + \lambda)x + |\lambda|^2)$ has only real coefficients. Then by the division algorithm, q also has only real coefficients.
- Now the result follows by simple induction. ■

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.
- Proof: Since the eigenvalues of A appear in conjugate pairs and n is odd it follows that A has a real eigenvalue.

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.
- Proof: Since the eigenvalues of A appear in conjugate pairs and n is odd it follows that A has a real eigenvalue.
- If λ is a real eigenvalue of a real matrix then we know that its geometric multiplicity is at least one. $\blacksquare$

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.
- Proof: Since the eigenvalues of A appear in conjugate pairs and n is odd it follows that A has a real eigenvalue.
- If λ is a real eigenvalue of a real matrix then we know that its geometric multiplicity is at least one. ■
- It maybe noted that existence of a real root can also be got from real analysis:

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.
- Proof: Since the eigenvalues of A appear in conjugate pairs and n is odd it follows that A has a real eigenvalue.
- If λ is a real eigenvalue of a real matrix then we know that its geometric multiplicity is at least one. ■
- It maybe noted that existence of a real root can also be got from real analysis:
- As the characteristic polynomial p is a monic real polynomial of odd degree

$$\lim_{x \to \infty} p(x) = +\infty; \quad \lim_{x \to -\infty} p(x) = -\infty$$

and so by intermediate value theorem p has at least one real root.

Continuation

- Corollary 16.12: Suppose A is a real $n \times n$ matrix. If n is odd then A has at least one real eigenvalue with real eigenvector.
- Proof: Since the eigenvalues of A appear in conjugate pairs and n is odd it follows that A has a real eigenvalue.
- If λ is a real eigenvalue of a real matrix then we know that its geometric multiplicity is at least one. ■
- It maybe noted that existence of a real root can also be got from real analysis:
- As the characteristic polynomial p is a monic real polynomial of odd degree

$$\lim_{x\to\infty} p(x) = +\infty; \quad \lim_{x\to-\infty} p(x) = -\infty$$

  and so by intermediate value theorem p has at least one real root.
- END OF LECTURE 16.