# LINEAR ALGEBRA -II

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# Lecture 17: Diagonalization

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- How to compute $F_{1000}$?
- Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Continuation

- We have:

$$A\begin{pmatrix}1\\0\end{pmatrix}=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{pmatrix}1\\1\end{pmatrix}.$$

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- Hence we know $F_{1000}$ if we know $A^{999}$.

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- Observation: Suppose there exists an invertible matrix $S$ such that $S^{-1}AS = D$ for some diagonal matrix $D$.
- Then
$$A = SDS^{-1}.$$
- This implies $A^2 = SD^2S^{-1}$ and more generally,
$$A^m = SD^mS^{-1}, \quad \forall m \geq 1.$$

# Continuation

- Now if

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

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- Hence computing $A^m$ becomes easy.

# Continuation

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- Suppose $A = SDS^{-1}$ with $D$ diagonal. What can be the diagonal entries?
- Let $p$ be the characteristic polynomial of $A$. From $A = SDS^{-1}$, we know that the characteristic polynomial of $A$ is same as that of $D$. Hence
$$p(x) = (x - d_1)(x - d_2) \cdots (x - d_n).$$

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- In particular the diagonal entries of $D$ must be the eigenvalues of $A$.

# Exmple

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is not diagonalizable.
- The eigenvalues of $B$ are just 0 and 0.
- So if $B = SDS^{-1}$, the diagonal $D$ must be the zero matrix.
- That would mean that $B = 0$, which is clearly not true. This is a contradiction. Hence $B$ is not diagonalizable.

# Linear independence

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- Theorem 17.3: Let $A$ be an $n \times n$ complex matrix. Suppose $a_1, \dots, a_k$ are some distinct eigenvalues of $A$ and $w_1, \dots, w_k$ are eigenvectors with

$$Aw_j = a_j w_j, \quad 1 \leq j \leq k.$$

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- Proof: Suppose $\sum_{j=1}^{k} c_j w_j = 0$.
- On applying $A$, we get $\sum_{j=1}^{k} a_j c_j w_j = 0$.
- By repeated application of $A$ we get

$$\sum_{j=1}^{k} a_j^s c_j w_j = 0, \forall 1 \leq s \leq (k-1).$$

# Continuation

- Let $N$ be the $n \times k$ matrix formed by taking the vectors $c_1 w_1, \ldots, c_k w_k$ as its columns:

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- Now we may write the linear equations above as:

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- Multiplying by $V^{-1}$ from the right, we get $N = 0$ or $c_j w_j = 0$ for every $j$. Since $w_j$'s are assumed to be non-zero (they are eigenvectors), we get $c_j = 0$ for every $j$. ■

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- Challenge: Find a different proof of this result.

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- (ii) There exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.
- (iii) The geometric multiplicity is same as the algebraic multiplicity for every eigenvalue of $A$.

# Continuation

- Proof: $(i) \Leftrightarrow (ii)$. The equation $A = SDS^{-1}$ is same as $AS = SD$. Let $v_1, v_2, \ldots, v_n$ be the columns of $S$.

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- Note that the invertibility of $S$ is equivalent to requiring that its columns $\{v_1, v_2, \ldots, v_n\}$ forms a basis of $\mathbb{C}^n$.

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- This proves $(i) \Leftrightarrow (ii)$.

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- (i) and $(ii) \Rightarrow (iii)$. From $A = SDS^{-1}$, we know that the characteristic polynomial of $A$ is same as that of $D$ and hence the eigenvalues of $A$ are $d_1, \ldots, d_n$ (including multiplicity).

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- Suppose $a_1, \ldots, a_k$ are distinct eigenvalues of $A$ and $r_j$ is the algebraic multiplicity of $a_j$, $1 \leq j \leq k$.

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- Suppose $a_1, \dots, a_k$ are distinct eigenvalues of $A$ and $r_j$ is the algebraic multiplicity of $a_j$, $1 \leq j \leq k$.
- Then taking a suitable permutation if necessary, we may assume that

$$(d_1, d_2, \dots, d_n) = (a_1, a_1, \dots, a_1, a_2, \dots, a_2, a_3, \dots a_k)$$

where $a_j$ appears $r_j$ times, $1 \leq j \leq k$ and
$r_1 + r_2 + \cdots + r_k = n$.

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- From $Av_j = d_j v_j$ and the fact that $\{v_1, \dots v_n\}$ are linearly independent, we see that the geometric multiplicity of $a_j$ is at least $r_j$.

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- It can't be more than $r_j$ as we have proved that the geometric multiplicity of any eigenvalue is less than or equal to its algebraic multiplicity.
- Therefore for every $j$, the geometric and algebraic multiplicity of $a_j$ is $r_j$. This proves (iii).

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- $(iii) \Rightarrow (ii)$. Let $a_1, a_2, \ldots a_k$ be the distinct eigenvalues of $A$ and let the geometric/algebraic multiplicity of $a_j$ be $r_j$.

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- By the fundamental theorem of algebra we know that $r_1 + \cdots + r_k = n$.
- Let $\{v_{j1}, v_{j2}, \dots, v_{jr_j}\}$ be a basis for the eigenspace of $A$ with eigenvalue $a_j$. In particular, for every $j$, $\{v_{j1}, v_{j2}, \dots, v_{jr_j}\}$ are linearly independent.

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- We also have $Av_{ji} = a_j v_{ji}$ for $1 \leq i \leq r_j$.

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- Take $w_j = \sum_{i=1}^{r_j} c_{ji} v_{ji}$.
- Note that $\sum_{j=1}^{k} w_j = 0$.
- Also $a_1, a_2, \ldots, a_k$ are distinct and $Aw_j = a_j w_j, 1 \leq j \leq k$.

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$$\{v_{ji} : 1 \leq i \leq r_j; 1 \leq j \leq k\}$$

  is linearly independent.

- Suppose

$$\sum_{j=1}^{k} \sum_{i=1}^{r_j} c_{ji} v_{ji} = 0$$

- Take $w_j = \sum_{i=1}^{r_j} c_{ji} v_{ji}$.
- Note that $\sum_{j=1}^{k} w_j = 0$.
- Also $a_1, a_2, \ldots, a_k$ are distinct and $Aw_j = a_j w_j, 1 \leq j \leq k$.
- Then by the previous theorem, $w_j = 0$ for every $j$.

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- For fixed $j$, by the linear independence of $v_{ji}$'s , we get $c_{ji} = 0$ for all $i$.
- This proves the required linear independence. ■

# Application

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- END OF LECTURE 17.