# LINEAR ALGEBRA -II

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# Lecture 18: Linear Recurrence relations

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- How to compute $F_{1000}$?
- Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Continuation

- We have:

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

# Continuation

- We have:

$$A\left(\begin{array}{c}1\\0\end{array}\right)=\left[\begin{array}{cc}1&1\\1&0\end{array}\right]\left[\begin{array}{c}1\\0\end{array}\right]=\left(\begin{array}{c}1\\1\end{array}\right).$$

- Further,

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- More generally,

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- Hence we know $F_{1000}$ if we know $A^{999}$.

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- Observation: Suppose there exists an invertible matrix $S$ such that $S^{-1}AS = D$ for some diagonal matrix $D$.
- Then
$$A = SDS^{-1}.$$
- This implies $A^2 = SD^2S^{-1}$ and more generally,
$$A^m = SD^mS^{-1}, \quad \forall m \geq 1.$$

# Continuation

- Now if

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

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- Hence computing $A^m$ becomes easy.

# Continuation

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- Suppose $A = SDS^{-1}$ with $D$ diagonal. What can be the diagonal entries?
- Let $p$ be the characteristic polynomial of $A$. From $A = SDS^{-1}$, we know that the characteristic polynomial of $A$ is same as that of $D$. Hence
$$p(x) = (x - d_1)(x - d_2) \cdots (x - d_n).$$

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- In particular the diagonal entries of $D$ must be the eigenvalues of $A$.

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- (ii) There exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.
- (iii) The geometric multiplicity is same as the algebraic multiplicity for every eigenvalue of $A$.

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- Proof: $(i) \Leftrightarrow (ii)$. The equation $A = SDS^{-1}$ is same as $AS = SD$. Let $v_1, v_2, \ldots, v_n$ be the columns of $S$.

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$$[Av_1, Av_2, \ldots, Av_n] = [d_1 v_1, d_2 v_2, \ldots, d_n v_n].$$

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- This proves $(i) \Leftrightarrow (ii)$.

# Summary

- We understand that to diagonalize a matrix we need to find eigenvalues and eigenvectors of the matrix and we need to get a basis consisting of eigenvectors.

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- We now apply this idea to solve some linear recurrence relations.

# Linear recurrence relations

- Suppose $a_0, a_1, \ldots, a_n, \ldots$ is a sequence of real/complex numbers defined by

$$a_0 = v_0, a_1 = v_1$$

and

$$a_n = ba_{n-1} + ca_{n-2}, \quad \forall n \geq 2$$

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- Therefore,

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} v_1 \\ v_0 \end{pmatrix}.$$

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- Case I: $\alpha \neq \beta$, that is, $b^2 + 4c \neq 0$.

# Continuation

- By direct computation we obtain eigenvectors

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$$S = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}.$$

- Then

$$S^{-1} = \frac{1}{\alpha - \beta} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}.$$

# Continuation

- From $A = SDS^{-1}$, we have

$$A = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \cdot \frac{1}{\alpha - \beta} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}.$$

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- Hence for $n \geq 1$,

$$\begin{aligned} A^{n-1} &= \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha^{n-1} & 0 \\ 0 & \beta^{n-1} \end{bmatrix} \cdot \frac{1}{\alpha - \beta} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^{n-1} & -\alpha^{n-1}\beta \\ -\beta^{n-1} & \alpha\beta^{n-1} \end{bmatrix} \\ &= \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^n - \beta^n & \alpha\beta^n - \alpha^n\beta \\ \alpha^{n-1} - \beta^{n-1} & \alpha\beta^{n-1} - \alpha^{n-1}\beta \end{bmatrix}. \end{aligned}$$

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- Therefore,

$$\begin{aligned} a_n &= \frac{1}{\alpha - \beta}[(\alpha^n - \beta^n)v_1 + (\alpha\beta^n - \alpha^n\beta)v_0] \\ &= \frac{1}{\alpha - \beta}[(v_1 - \beta v_0)\alpha^n + (\alpha v_0 - v_1)\beta^n]. \end{aligned}$$

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$$\alpha = \frac{b+\sqrt{b^2+4c}}{2}, \quad \beta = \frac{b-\sqrt{b^2+4c}}{2}.$$

- Therefore, $\alpha-\beta=\sqrt{b^2+4c}$.
- In particular, $a_n$ has the form

$$a_n = s\alpha^n + t\beta^n$$

for some scalars $s, t$, where $\alpha, \beta$ are the two distinct roots of $x^2-bx-c=0$.

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and $\alpha - \beta = \sqrt{5}.$

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and $\alpha - \beta = \sqrt{5}$.
- Therefore,
$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n], \quad \forall n \geq 0.$$

# Example

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- So the formula for $a_n$ has the form

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- Though all the terms of the sequence are real, the formula for $a_n$ requires complex terms!

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- What about higher order recurrence relations such as tribonacci numbers:

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- Here

$$t_0 = t_1 = 0, t_2 = 1, t_n = t_{n-1} + t_{n-2} + t_{n-3}, \quad \forall n \geq 3.$$

# Continuation

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- END OF LECTURE 18.