LINEAR ALGEBRA -II

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- Recall:

Lecture 19: Schur's upper triangularization theorem

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- (ii) There exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
- (iii) The geometric multiplicity is same as the algebraic multiplicity for every eigenvalue of A.
- There are matrices which are not diagonalizable. The next best would be to make the matrix 'triangular'.

Upper and lower triangular matrices

- Definition 19.1: A matrix $T = [t_{ij}]_{1 \leq i,j \leq n}$ is said to be upper triangular if
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- Upper triangular:
$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & t_{22} & t_{23} & \dots & t_{2n} \\ 0 & 0 & t_{33} & \dots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{bmatrix}.$$
- Note that products of upper triangular matrices are upper triangular. If a matrix is both upper triangular and lower triangular then it is diagonal.

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- For $n = 1$ there is nothing to prove, as every 1×1 matrix is upper triangular, we can take U as the 1×1 identity matrix.
- Now take $n \geq 2$ and assume the result for all $(n-1) \times (n-1)$ matrices.

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- Let v_1 be an eigenvector of A with eigenvalue a_1. (Since $\det(A - a_1 I) = 0$, $A - a_1 I$ is singular and so there exists a non-zero vector v_1 such that $(A - a_1 I)v_1 = 0$.)

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- By dividing v_1 by its norm if necessary, we may assume that v_1 is a unit vector.

Continuation

- Extend $\{v_1\}$ to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$. (Such an extension exists due to Gram-Schmidt orthogonalization process.)

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- We have $Av_1 = a_1 v_1$ and for every j, expanding Av_j using the basis $\{v_1, \ldots, v_n\}$,:

$$Av_j = \sum_{i=1}^{n} \langle v_i, Av_j \rangle v_i.$$

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$$AV = VS$$

where $S = [s_{ij}]$ is the matrix defined by

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- In other words, S is the matrix of the linear map $x \mapsto Ax$, on the basis $\{v_1, \ldots, v_n\}$.

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- Note that since $Av_1 = a_1v_1$ and $s_{ij} = \langle v_i, Av_j \rangle$, the matrix S is of the form:
$$S = \begin{bmatrix} a_1 & y \\ 0 & B \end{bmatrix}$$
for some $1 \times (n-1)$ vector y and $(n-1) \times (n-1)$ matrix B.

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- By induction hypothesis, there exists an $(n-1) \times (n-1)$ unitary matrix U_1 and an upper triangular matrix T_1 such that

$$B = U_1 T_1 U_1^*.$$

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- We may re-write this as:

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- This completes the proof.

Diagonal entries

- Remark 19.2: Suppose A is an $n \times n$ matrix, U is a unitary and T is an $n \times n$ upper triangular matrix such that $A = UTU^*$. Then the charactristic polynomials of A and T are same. Further, diagonal entries of T are eigenvalues of A.

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- AS A and T are similar they have same characteristic polynomial.
- The second part follows as determinant of any upper triangular matrix is product of its diagonal entries and hence

$$\det(I - A) = \det(xI - T) = (x - t_{11})(x - t_{22}) \cdots (x - t_{nn}).$$

Linear recurrence relations

- Recall: Suppose $a_0, a_1, \ldots, a_n, \ldots$ is a sequence of real/complex numbers defined by

$$a_0 = v_0, a_1 = v_1$$

and

$$a_n = ba_{n-1} + ca_{n-2}, \quad \forall n \geq 2$$

where v_0, v_1, b, c are some complex numbers.

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$$A \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$
- Therefore,
$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} v_1 \\ v_0 \end{pmatrix}.$$

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- Case (ii): $b^2 + 4c = 0$. So the two roots are equal to $\frac{b}{2}$.

Linear recurrence relation with repeated roots

- Consider the matrix
$$A = \begin{bmatrix} b & c \\ 1 & 0 \end{bmatrix}.$$
where $b^2 + 4c = 0$ and so the eigenvalues of A are $\frac{b}{2}$ and $\frac{b}{2}$.

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- Now
$$\left(\begin{array}{c} \frac{b}{2} \\ 1 \end{array} \right)$$
is an eigenvector with eigenvalue $\frac{b}{2}$. Up to scaling this is the only eigenvector.
- Further
$$\left(\begin{array}{c} -1 \\ \frac{b}{2} \end{array} \right)$$
is a vector orthogonal to
$$\left(\begin{array}{c} \frac{b}{2} \\ 1 \end{array} \right).$$

Continuation

- Normalizing these vectors we get an orthonormal basis $\{u_1, u_2\}$ for $\mathbb{C}^2$ where

$$u_1 = \frac{1}{d}\begin{pmatrix} \frac{b}{2} \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{d}\begin{pmatrix} -1 \\ \frac{\bar{b}}{2} \end{pmatrix}$$

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  for a unitary U with T upper-triangular.
- By comparing eigenvalues of A and T,

$$T = \begin{bmatrix} \frac{b}{2} & p \\ 0 & \frac{b}{2} \end{bmatrix}$$

  for some p.

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- It is easy to see from induction that

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for some scalars s, t. (Do the necessary matrix computations to verify this.)

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- The scalars can be determined using the initial conditions $a_0 = v_0$ and $a_1 = v_1$.

Example

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  Hence $t = -\frac{1}{3}$.

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- Taking $n = 0, 1$ we get
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Hence $t = -\frac{1}{3}$.
- Therefore
$$a_n = 3^n(1 - \frac{n}{3}), \quad \forall n \geq 0.$$
- END OF LECTURE 19.