LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 20: Normal matrices

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- The diagonal entries of D are eigenvalues of A and columns of S are corresponding eigenvectors.

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- (ii) There exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
- (iii) The geometric multiplicity is same as the algebraic multiplicity for every eigenvalue of A.
- There are matrices which are not diagonalizable. The next best would be to make the matrix 'triangular'.

Upper and lower triangular matrices

- Definition 19.1: A matrix $T = [t_{ij}]_{1 \leq i,j \leq n}$ is said to be upper triangular if

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- Upper triangular:
$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & t_{22} & t_{23} & \dots & t_{2n} \\ 0 & 0 & t_{33} & \dots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{bmatrix}.$$
- Note that products of upper triangular matrices are upper triangular. If a matrix is both upper triangular and lower triangular then it is diagonal.

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- AS A and T are similar they have same characteristic polynomial.
- The second part follows as determinant of any upper triangular matrix is product of its diagonal entries and hence

$$\det(xI - A) = \det(xI - T) = (x - t_{11})(x - t_{22}) \cdots (x - t_{nn}).$$

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- We have seen that computations can be made easily with diagonal matrices.
- However, in general it is very difficult to check diagonalizability.
- So we focus on a large class of matrices called normal matrices. Normality is easy to check and it ensures diagonalizability.

Self-adjoint and normal matrices

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- In particular, every real symmetric matrix is self-adjoint.
- Here is an example of a self-adjoint matrix which is not real and symmetric:

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- Note that diagonal entry of every self-adjoint matrix is real as $\overline{a_{ii}} = a_{ii}$ for every i.

Examples of normal matrices

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- (iv) Every diagonal matrix is normal. Every real diagonal matrix is self-adjoint.
- Example 20.3: Consider
$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Then C is not normal.

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- We have

$$T^* = \begin{bmatrix} \overline{t_{11}} & 0 & 0 & \dots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & 0 & \dots & 0 \\ \overline{t_{13}} & \overline{t_{23}} & \overline{t_{33}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{t_{1n}} & \overline{t_{2n}} & \overline{t_{3n}} & \dots & \overline{t_{nn}} \end{bmatrix}$$

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- Computing the first diagonal entries of T^*T and TT^*, as T is normal, we get

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- So we get

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- In other words, T is diagonal. ■

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- Proposition 20.6: Suppose B is unitarily equivalent to A. Then B is normal (resp. self-adjoint, unitary, projection) if and only if A is normal (resp. self-adjoint, unitary, projection).
- Proof: Suppose U is a unitary such that $B = UAU^*$. Then $B^*B = (UAU^*)^*(UAU^*) = UA^*UU^*AU = UA^*AU^*$. Similarly, $BB^* = UAA^*U^*$. Now the result follows easily.

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- Note that A and T are unitarily equivalent. Consequently T is normal. Then by Theorem 20.4, as T is both upper triangular and normal it must be diagonal. Taking $D = T$, we have $A = UDU^*$ and we are done.

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