LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

- Recall: We recall some definitions and the spectral theorem for normal matrices.

Self-adjoint and normal matrices

- Definition 20.1: (i) A complex square matrix A is said to be self-adjoint if $A^* = A$. (ii) A complex square matrix A is said to be normal if $A^*A = AA^*$.

Self-adjoint and normal matrices

- Definition 20.1: (i) A complex square matrix A is said to be self-adjoint if $A^* = A$. (ii) A complex square matrix A is said to be normal if $A^*A = AA^*$.
- Recall that for $A = [a_{ij}]_{1 \leq i,j \leq n}$, we have $A^* = [\overline{a_{ji}}]_{1 \leq i,j \leq n}$.

Self-adjoint and normal matrices

- Definition 20.1: (i) A complex square matrix A is said to be self-adjoint if $A^* = A$. (ii) A complex square matrix A is said to be normal if $A^*A = AA^*$.
- Recall that for $A = [a_{ij}]_{1 \leq i,j \leq n}$, we have $A^* = [\overline{a_{ji}}]_{1 \leq i,j \leq n}$.
- In particular, every real symmetric matrix is self-adjoint.
- Here is an example of a self-adjoint matrix which is not real and symmetric:

$$B = \begin{bmatrix} 2 & 3+5i \\ 3-5i & 1 \end{bmatrix}.$$

Self-adjoint and normal matrices

- Definition 20.1: (i) A complex square matrix A is said to be self-adjoint if $A^* = A$. (ii) A complex square matrix A is said to be normal if $A^*A = AA^*$.
- Recall that for $A = [a_{ij}]_{1 \leq i,j \leq n}$, we have $A^* = [\overline{a_{ji}}]_{1 \leq i,j \leq n}$.
- In particular, every real symmetric matrix is self-adjoint.
- Here is an example of a self-adjoint matrix which is not real and symmetric:

$$B = \begin{bmatrix} 2 & 3+5i \\ 3-5i & 1 \end{bmatrix}.$$

- Note that diagonal entry of every self-adjoint matrix is real as $\overline{a_{ii}} = a_{ii}$ for every i.

Examples of normal matrices

- Remark 20.2: (i) If A is self-adjoint then A is normal as

$$A^* A = A.A = AA^*.$$

Examples of normal matrices

- Remark 20.2: (i) If A is self-adjoint then A is normal as

$$A^*A = A.A = AA^*.$$

- (ii) If A is a unitary then A is normal as

$$AA^* = A^*A = I.$$

Examples of normal matrices

- Remark 20.2: (i) If A is self-adjoint then A is normal as

$$A^*A = A.A = AA^*.$$

- (ii) If A is a unitary then A is normal as

$$AA^* = A^*A = I.$$

- (iii) If A is a projection then $A = A^* = A^2$. Hence A is self-adjoint and normal.

Examples of normal matrices

- Remark 20.2: (i) If A is self-adjoint then A is normal as

$$A^*A = A.A = AA^*.$$

- (ii) If A is a unitary then A is normal as

$$AA^* = A^*A = I.$$

- (iii) If A is a projection then $A = A^* = A^2$. Hence A is self-adjoint and normal.
- (iv) Every diagonal matrix is normal. Every real diagonal matrix is self-adjoint.

Examples of normal matrices

- Remark 20.2: (i) If A is self-adjoint then A is normal as
$$A^*A = A.A = AA^*.$$
- (ii) If A is a unitary then A is normal as
$$AA^* = A^*A = I.$$
- (iii) If A is a projection then $A = A^* = A^2$. Hence A is self-adjoint and normal.
- (iv) Every diagonal matrix is normal. Every real diagonal matrix is self-adjoint.
- Example 20.3: Consider
$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Then C is not normal.

Normal upper triangular matrices

- Theorem 20.4: If a square matrix is upper triangular and normal then it is diagonal.

Normal upper triangular matrices

- Theorem 20.4: If a square matrix is upper triangular and normal then it is diagonal.
- Proof: Suppose

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & t_{22} & t_{23} & \dots & t_{2n} \\ 0 & 0 & t_{33} & \dots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

is normal.

Normal upper triangular matrices

- Theorem 20.4: If a square matrix is upper triangular and normal then it is diagonal.
- Proof: Suppose

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots & t_{1n} \\ 0 & t_{22} & t_{23} & \dots & t_{2n} \\ 0 & 0 & t_{33} & \dots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

is normal.
- We have

$$T^* = \begin{bmatrix} \overline{t_{11}} & 0 & 0 & \dots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & 0 & \dots & 0 \\ \overline{t_{13}} & \overline{t_{23}} & \overline{t_{33}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{t_{1n}} & \overline{t_{2n}} & \overline{t_{3n}} & \dots & \overline{t_{nn}} \end{bmatrix}$$

.

Continuation

- Computing the first diagonal entries of T^*T and TT^*, as T is normal, we get

$$|t_{11}|^2 = |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2.$$

Continuation

- Computing the first diagonal entries of T^*T and TT^*, as T is normal, we get

$$|t_{11}|^2 = |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2.$$

- Hence $|t_{12}|^2 + \cdots + |t_{1n}|^2 = 0$ or

$$t_{12} = t_{13} = \cdots = t_{1n} = 0.$$

Continuation

- Computing the first diagonal entries of T^*T and TT^*, as T is normal, we get

$$|t_{11}|^2 = |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2.$$

- Hence $|t_{12}|^2 + \cdots + |t_{1n}|^2 = 0$ or

$$t_{12} = t_{13} = \cdots = t_{1n} = 0.$$

- So we get

$$T = \begin{bmatrix} t_{11} & 0 & 0 & \dots & 0 \\ 0 & t_{22} & t_{23} & \dots & t_{2n} \\ 0 & 0 & t_{33} & \dots & t_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

Continuation

Continuation

- Now from $(T^*T)_{22} = (TT^*)_{22}$ we get

$$|t_{22}|^2 = |t_{22}|^2 + |t_{23}|^2 + \cdots + |t_{2n}|^2.$$

Continuation

- Now from $(T^*T)_{22} = (TT^*)_{22}$ we get

$$|t_{22}|^2 = |t_{22}|^2 + |t_{23}|^2 + \cdots + |t_{2n}|^2.$$

- Consequently,

$$t_{23} = t_{24} = \cdots = t_{2n} = 0.$$

Continuation

- Now from $(T^*T)_{22} = (TT^*)_{22}$ we get

$$|t_{22}|^2 = |t_{22}|^2 + |t_{23}|^2 + \cdots + |t_{2n}|^2.$$

- Consequently,

$$t_{23} = t_{24} = \cdots = t_{2n} = 0.$$

- Continuing this way (that is, by mathematical induction) we see that $t_{ij} = 0, \quad \forall i \neq j$.

Continuation

- Now from $(T^*T)_{22} = (TT^*)_{22}$ we get

$$|t_{22}|^2 = |t_{22}|^2 + |t_{23}|^2 + \cdots + |t_{2n}|^2.$$

- Consequently,

$$t_{23} = t_{24} = \cdots = t_{2n} = 0.$$

- Continuing this way (that is, by mathematical induction) we see that $t_{ij} = 0, \quad \forall i \neq j$.
- In other words, T is diagonal. ■

Unitary equivalence

- Definition 20.5: A complex matrix B is said to be unitarily equivalent to a matrix A if there exists a unitary matrix U such that
$$B = UAU^*.$$

Unitary equivalence

- Definition 20.5: A complex matrix B is said to be unitarily equivalent to a matrix A if there exists a unitary matrix U such that
$$B = UAU^*.$$
- It is easy to see that unitary equivalence is an equivalence relation in the space of matrices.

Unitary equivalence

- Definition 20.5: A complex matrix B is said to be unitarily equivalent to a matrix A if there exists a unitary matrix U such that
$$B = UAU^*.$$
- It is easy to see that unitary equivalence is an equivalence relation in the space of matrices.
- B is unitarily equivalent to A means that A, B are matrices of same linear map on different orthonormal bases.

Unitary equivalence

- Definition 20.5: A complex matrix B is said to be unitarily equivalent to a matrix A if there exists a unitary matrix U such that
$$B = UAU^*.$$
- It is easy to see that unitary equivalence is an equivalence relation in the space of matrices.
- B is unitarily equivalent to A means that A, B are matrices of same linear map on different orthonormal bases.
- Proposition 20.6: Suppose B is unitarily equivalent to A. Then B is normal (resp. self-adjoint, unitary, projection) if and only if A is normal (resp. self-adjoint, unitary, projection).

Unitary equivalence

- Definition 20.5: A complex matrix B is said to be unitarily equivalent to a matrix A if there exists a unitary matrix U such that
$$B = UAU^*.$$
- It is easy to see that unitary equivalence is an equivalence relation in the space of matrices.
- B is unitarily equivalent to A means that A, B are matrices of same linear map on different orthonormal bases.
- Proposition 20.6: Suppose B is unitarily equivalent to A. Then B is normal (resp. self-adjoint, unitary, projection) if and only if A is normal (resp. self-adjoint, unitary, projection).
- Proof: Suppose U is a unitary such that $B = UAU^*$. Then $B^*B = (UAU^*)^*(UAU^*) = UA^*UU^*AU = UA^*AU^*$. Similarly, $BB^* = UAA^*U^*$. Now the result follows easily.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*$$
if and only if A is normal.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that

$$A = UDU^*$$

  if and only if A is normal.
- The theorem tells us that a matrix is diagonalizable through a unitary iff it is normal. In particular, every normal matrix is diagonalizable.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*$$
if and only if A is normal.
- The theorem tells us that a matrix is diagonalizable through a unitary iff it is normal. In particular, every normal matrix is diagonalizable.
- Proof: Suppose A is a normal matrix.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*$$
if and only if A is normal.
- The theorem tells us that a matrix is diagonalizable through a unitary iff it is normal. In particular, every normal matrix is diagonalizable.
- Proof: Suppose A is a normal matrix.
- By Schur's upper triangularization theorem (Theorem 19.2) there exists a unitary U and an upper triangular matrix T such that
$$A = UTU^*.$$

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*$$
if and only if A is normal.
- The theorem tells us that a matrix is diagonalizable through a unitary iff it is normal. In particular, every normal matrix is diagonalizable.
- Proof: Suppose A is a normal matrix.
- By Schur's upper triangularization theorem (Theorem 19.2) there exists a unitary U and an upper triangular matrix T such that
$$A = UTU^*.$$
- Note that A and T are unitarily equivalent. Consequently T is normal. Then by Theorem 20.4, as T is both upper triangular and normal it must be diagonal. Taking $D = T$, we have $A = UDU^*$ and we are done.

Continuation

- Conversely, suppose $A = UDU^*$ where D is diagonal and U is a unitary.

Continuation

- Conversely, suppose $A = UDU^*$ where D is diagonal and U is a unitary.
- Since every diagonal matrix is normal, D is normal.

Continuation

- Conversely, suppose $A = UDU^*$ where D is diagonal and U is a unitary.
- Since every diagonal matrix is normal, D is normal.
- Then as A is unitarily equivalent to D, A is also normal. ■.

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.
- Proof: Suppose A is normal. Then by the spectral theorem there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*.$$

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.
- Proof: Suppose A is normal. Then by the spectral theorem there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*.$$
- Writing this as $AU = UD$, we see that columns of U are eigenvectors for A. Since U is a unitary its columns form an orthonormal basis for $\mathbb{C}^n$.

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.
- Proof: Suppose A is normal. Then by the spectral theorem there exists a unitary matrix U and a diagonal matrix D such that

$$A = UDU^*.$$

- Writing this as $AU = UD$, we see that columns of U are eigenvectors for A. Since U is a unitary its columns form an orthonormal basis for $\mathbb{C}^n$.
- Conversely, suppose $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A, say $Av_j = d_j v_j, 1 \leq j \leq n$.

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.
- Proof: Suppose A is normal. Then by the spectral theorem there exists a unitary matrix U and a diagonal matrix D such that
$$A = UDU^*.$$
- Writing this as $AU = UD$, we see that columns of U are eigenvectors for A. Since U is a unitary its columns form an orthonormal basis for $\mathbb{C}^n$.
- Conversely, suppose $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A, say $Av_j = d_j v_j, 1 \leq j \leq n$.
- Take $U = [v_1, \ldots, v_n]$. Then U is a unitary and $AU = UD$. Hence $A = UDU^*$. Consequently A is normal. ■

Example

- Example 21.2: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Example

- Example 21.2: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Then as A is self-adjoint it is normal.

Example

- Example 21.2: Suppose
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$
- Then as A is self-adjoint it is normal.
- The characteristic polynomial of A is
$$p(x) = \det(xI - A) = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-3)(x-1).$$

Example

- Example 21.2: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Then as A is self-adjoint it is normal.
- The characteristic polynomial of A is

$$p(x) = \det(xI - A) = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-3)(x-1).$$

- Hence the eigenvalues of A are 1 and 3.

Example

- Example 21.2: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

- Then as A is self-adjoint it is normal.
- The characteristic polynomial of A is

$$p(x) = \det(xI - A) = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-3)(x-1).$$

- Hence the eigenvalues of A are 1 and 3.
- Solving corresponding eigen equations we see that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Continuation

- Normalizing these eigenvectors, and taking them as columns we get a unitary,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Continuation

- Normalizing these eigenvectors, and taking them as columns we get a unitary,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- satisfying,

$$AU = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Continuation

- Normalizing these eigenvectors, and taking them as columns we get a unitary,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- satisfying,

$$AU = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Alternatively,

$$A = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} U^*.$$

Terminology and notation

- Definition 21.3: Let A be a complex square matrix. Then

$$\sigma(A) = \{c \in \mathbb{C} : (cI - A) \text{ is not invertible}\}$$

is known as the spectrum of A.

Terminology and notation

- Definition 21.3: Let A be a complex square matrix. Then

$$\sigma(A) = \{c \in \mathbb{C} : (cI - A) \text{ is not invertible}\}$$

is known as the spectrum of A.

- Note that for a matrix A, if $a_1, a_2, \ldots, a_n$ are eigenvalues of A, then

$$\sigma(A) = \{a_1, a_2, \ldots, a_n\}$$

Some characterizations

- Theorem 21.4: Let A be a normal matrix. Then,
  (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
  (ii) A is a unitary iff $\sigma(A) \subset \{z \in \mathbb{C} : |z| = 1\}$.
  (iii) A is a projection iff $\sigma(A) \subseteq \{0, 1\}$.

Some characterizations

- Theorem 21.4: Let A be a normal matrix. Then,
  (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
  (ii) A is a unitary iff $\sigma(A) \subset \{z \in \mathbb{C} : |z| = 1\}$.
  (iii) A is a projection iff $\sigma(A) \subseteq \{0, 1\}$.
- Proof: By spectral theorem

$$A = UDU^*$$

for some unitary U and a diagonal matrix D. By taking adjoint,

$$A^* = (UDU^*)^* = UD^*U^*.$$

Some characterizations

- Theorem 21.4: Let A be a normal matrix. Then,
  (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
  (ii) A is a unitary iff $\sigma(A) \subset \{z \in \mathbb{C} : |z| = 1\}$.
  (iii) A is a projection iff $\sigma(A) \subseteq \{0, 1\}$.
- Proof: By spectral theorem

$$A = UDU^*$$

for some unitary U and a diagonal matrix D. By taking adjoint,

$$A^* = (UDU^*)^* = UD^*U^*.$$

- If A is self-adjoint, then $A = A^*$. Hence,

$$UDU^* = UD^*U^*.$$

Some characterizations

- Theorem 21.4: Let A be a normal matrix. Then,
  (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
  (ii) A is a unitary iff $\sigma(A) \subset \{z \in \mathbb{C} : |z| = 1\}$.
  (iii) A is a projection iff $\sigma(A) \subseteq \{0, 1\}$.
- Proof: By spectral theorem

$$A = UDU^*$$

  for some unitary U and a diagonal matrix D. By taking adjoint,

$$A^* = (UDU^*)^* = UD^*U^*.$$

- If A is self-adjoint, then $A = A^*$. Hence,

$$UDU^* = UD^*U^*.$$

- Multiplication by U^*, U, yields $D = D^*$. Since D is diagonal, this means that all the diagonal entries are real. Hence

$$\sigma(A) \subset \mathbb{R}.$$

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.
- Properties (ii) and (iii) also follow easily from the spectral theorem. ■

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.
- Properties (ii) and (iii) also follow easily from the spectral theorem. ■
- Remark 21.5: In this theorem the hypothesis of A being normal is crucial. Just from knowing that eigenvalues of A are real one can't say that it is self-adjoint.

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.
- Properties (ii) and (iii) also follow easily from the spectral theorem. ■
- Remark 21.5: In this theorem the hypothesis of A being normal is crucial. Just from knowing that eigenvalues of A are real one can't say that it is self-adjoint.
- For instance,

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix}$$

has $\{3, 7\}$ as its spectrum, which is a subset of the real line.

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.
- Properties (ii) and (iii) also follow easily from the spectral theorem. ■
- Remark 21.5: In this theorem the hypothesis of A being normal is crucial. Just from knowing that eigenvalues of A are real one can't say that it is self-adjoint.
- For instance,
$$A = \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix}$$
has $\{3, 7\}$ as its spectrum, which is a subset of the real line.
- However, clearly A is not self-adjoint.

Continuation

- Conversely, if the eigenvalues of A are real, then the diagonal entries of D are real and hence $D = D^*$. Now $A = A^*$ is clear.
- Properties (ii) and (iii) also follow easily from the spectral theorem. ■
- Remark 21.5: In this theorem the hypothesis of A being normal is crucial. Just from knowing that eigenvalues of A are real one can't say that it is self-adjoint.
- For instance,

$$A = \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix}$$

  has $\{3, 7\}$ as its spectrum, which is a subset of the real line.
- However, clearly A is not self-adjoint.
- Similarly (ii) and (iii) of this theorem do not hold without the assumption of normality.

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned}\|Ax\|^2 &= \langle Ax, Ax\rangle \\ &= \langle x, A^*Ax\rangle \\ &= \langle x, AA^*x\rangle \\ &= \langle A^*x.A^*x\rangle \\ &= \|A^*x\|^2.\end{aligned}$$

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax\rangle \\ &= \langle x, A^*Ax\rangle \\ &= \langle x, AA^*x\rangle \\ &= \langle A^*x.A^*x\rangle \\ &= \|A^*x\|^2. \end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= \langle x, A^*Ax \rangle \\ &= \langle x, AA^*x \rangle \\ &= \langle A^*x.A^*x \rangle \\ &= \|A^*x\|^2. \end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Conversely, suppose $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= \langle x, A^*Ax \rangle \\ &= \langle x, AA^*x \rangle \\ &= \langle A^*x . A^*x \rangle \\ &= \|A^*x\|^2. \end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Conversely, suppose $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Then $\langle x, A^*Ax \rangle = \langle x, AA^*x \rangle, \quad \forall x \in \mathbb{C}^n$.

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned}\|Ax\|^2 &= \langle Ax, Ax\rangle \\ &= \langle x, A^*Ax\rangle \\ &= \langle x, AA^*x\rangle \\ &= \langle A^*x.A^*x\rangle \\ &= \|A^*x\|^2.\end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Conversely, suppose $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Then $\langle x, A^*Ax\rangle = \langle x, AA^*x\rangle, \quad \forall x \in \mathbb{C}^n$.
- Polarization identity yields,

$$\langle x, A^*Ay\rangle = \langle x, AA^*y\rangle, \quad \forall x, y \in \mathbb{C}^n.$$

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax\rangle \\ &= \langle x, A^*Ax\rangle \\ &= \langle x, AA^*x\rangle \\ &= \langle A^*x.A^*x\rangle \\ &= \|A^*x\|^2. \end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Conversely, suppose $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Then $\langle x, A^*Ax\rangle = \langle x, AA^*x\rangle, \quad \forall x \in \mathbb{C}^n$.
- Polarization identity yields,

$$\langle x, A^*Ay\rangle = \langle x, AA^*y\rangle, \quad \forall x, y \in \mathbb{C}^n.$$

- Hence $A^*A = AA^*$. ■

Another characterization of a normal matrix

- Theorem 21.6: Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- Proof: Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= \langle x, A^*Ax \rangle \\ &= \langle x, AA^*x \rangle \\ &= \langle A^*x . A^*x \rangle \\ &= \|A^*x\|^2. \end{aligned}$$

- Hence $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Conversely, suppose $\|Ax\| = \|A^*x\|, \quad \forall x \in \mathbb{C}^n$.
- Then $\langle x, A^*Ax \rangle = \langle x, AA^*x \rangle, \quad \forall x \in \mathbb{C}^n$.
- Polarization identity yields,

$$\langle x, A^*Ay \rangle = \langle x, AA^*y \rangle, \quad \forall x, y \in \mathbb{C}^n.$$

- Hence $A^*A = AA^*$. ■
- END OF LECTURE 21