LINEAR ALGEBRA -II

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Lecture 22: Path counting

- Recall: We recall some definitions and the spectral theorem for normal matrices.

Self-adjoint and normal matrices

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- In particular, every real symmetric matrix is self-adjoint.
- Here is an example of a self-adjoint matrix which is not real and symmetric:

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- Note that diagonal entry of every self-adjoint matrix is real as $\overline{a_{ii}} = a_{ii}$ for every i.

Examples of normal matrices

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- (iv) Every diagonal matrix is normal. Every real diagonal matrix is self-adjoint.
- Example 20.3: Consider
$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

  Then C is not normal.

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- Computing the first diagonal entries of T^*T and TT^*, as T is normal, we get

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- In other words, T is diagonal. ■

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- Proposition 20.6: Suppose B is unitarily equivalent to A. Then B is normal (resp. self-adjoint, unitary, projection) if and only if A is normal (resp. self-adjoint, unitary, projection).
- Proof: Suppose U is a unitary such that $B = UAU^*$. Then $B^*B = (UAU^*)^*(UAU^*) = UA^*UU^*AU = UA^*AU^*$. Similarly, $BB^* = UAA^*U^*$. Now the result follows easily.

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- Proof: Suppose A is a normal matrix.
- By Schur's upper triangularization theorem (Theorem 19.2) there exists a unitary U and an upper triangular matrix T such that
$$A = UTU^*.$$
- Note that A and T are unitarily equivalent. Consequently T is normal. Then by Theorem 20.4, as T is both upper triangular and normal it must be diagonal. Taking $D = T$, we have $A = UDU^*$ and we are done.

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- Since every diagonal matrix is normal, D is normal.
- Then as A is unitarily equivalent to D, A is also normal. $\blacksquare$.

Consequences of the spectral theorem

- Corollary 21.1: Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ consisting of eigenvectors of A.

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- Take $U = [v_1, \ldots, v_n]$. Then U is a unitary and $AU = UD$. Hence $A = UDU^*$. Consequently A is normal. ■

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- Solving corresponding eigen equations we see that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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- Normalizing these eigenvectors, and taking them as columns we get a unitary,

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- Alternatively,

$$A = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} U^*.$$

Terminology and notation

- Definition 21.3: Let A be a complex square matrix. Then

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- Note that for a matrix A, if $a_1, a_2, \ldots, a_n$ are eigenvalues of A, then

$$\sigma(A) = \{a_1, a_2, \ldots, a_n\}$$

Some characterizations

- Theorem 21.4: Let A be a normal matrix. Then,
  (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
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- Multiplication by U^*, U, yields $D = D^*$. Since D is diagonal, this means that all the diagonal entries are real. Hence

$$\sigma(A) \subset \mathbb{R}.$$

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- Similarly (ii) and (iii) of this theorem do not hold without the assumption of normality.

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$$\langle x, A^*Ay \rangle = \langle x, AA^*y \rangle, \quad \forall x, y \in \mathbb{C}^n.$$

- Hence $A^*A = AA^*$. ■

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- In particular, $\|(A - cI)x\| = 0$ if and only if $\|(A - cI)^*x\| = 0$.
- As $(A - cI)^* = A^* - \bar{c}I$ this shows that x is an eigenvector for A with eigenvalue c if and only if it is an eigenvector for A^* with eigenvalue $\bar{c}$.

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- Exercise 22.4: Suppose A is a normal matrix. Then show that a matrix B commutes with A if and only if it commutes with A^*.

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- Examples of such paths:
- $1 \to 2 \to 4 \to 2 \to 4$, $1 \to 2 \to 1 \to 3 \to 4$.

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- What is the number of paths from 1 to 4 in 100 steps?
- How do we count such paths?

Continuation

- A graph G is a pair (V, E) where V is a set called the set of vertices and E is a collection of pairs of the form $\{x, y\}$ where $x, y \in V$, with $x \neq y$, called the edges of the graph.

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- For the square graph example above: $V = \{1, 2, 3, 4\}$ and

$$E = \{\{1, 2\}, \{2, 4\}, \{1, 3\}, \{3, 4\}\}.$$

Paths

- In the following, we take $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ and so E is a collection of pairs of the form $\{i, j\}$ with $i, j \in V$.

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- Note that k_j's need not be distinct.

Adjacency matrix

- Definition 22.6: The adjacency matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ of a graph $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is defined by taking

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

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$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

- Note that A is a real symmetric matrix and the diagonal entries are all equal to zero.

Two step paths

- We have

$$\begin{aligned}
(A^2)_{ij} &= \sum_{k=1}^{n} a_{ik} a_{kj} \\
&= \sum_{k: a_{ik}.a_{kj} \neq 0} a_{ik} a_{kj} \\
&= \sharp\{k : a_{ik}.a_{kj} \neq 0\} \\
&= \sharp\{k : a_{ik} \neq 0, \quad a_{kj} \neq 0\} \\
&= \sharp\{k : (i,k), (k,j) \in E\} \\
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&= \sharp\{k : (i,k),(k,j) \in E\} \\
&= \quad \text{The number of two step paths from } i \text{ to } j.
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- Therefore, $(i,j)^{\text{th}}$-entry of A^2 is the number of two step paths from (i,j).

m-step paths

- For $m \geq 2$,

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\begin{aligned}
& (A^m)_{ij} \\
= & \sum_{k_1, k_2, \ldots, k_{m-1}} a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \\
= & \sharp\{(k_1, k_2, \ldots, k_{m-1}) : a_{1k_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \neq 0\} \\
= & \sharp\{(i, k_1, \ldots, k_{m-1}, j) : (i, k_1), (k_1, k_2), \ldots, (k_{m-1}, j) \in E\} \\
= & \text{ Number of paths of length } m \text{ from } i \text{ to } j.
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m-step paths

- For $m \geq 2$,
- $$\begin{aligned} & (A^m)_{ij} \\ = & \sum_{k_1, k_2, \ldots, k_{m-1}} a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \\ = & \sharp\{(k_1, k_2, \ldots, k_{m-1}) : a_{1k_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \neq 0\} \\ = & \sharp\{(i, k_1, \ldots, k_{m-1}, j) : (i, k_1), (k_1, k_2), \ldots, (k_{m-1}, j) \in E\} \\ = & \text{ Number of paths of length } m \text{ from } i \text{ to } j. \end{aligned}$$
- In other words, $(i, j)^{\text{th}}$-entry of A^m is exactly the number of paths of length m from i to j in the graph G, where A is the adjacency matrix of the graph G.

Computation

- Observe that the adjacency matrix of a graph G is a real symmetric matrix. Hence it is normal and also has real eigenvalues.

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for some unitary U and real diagonal matrix D.
- Consequently $A^m = UD^mU^*$. This allows us to compute the number of paths of length m between any to vertices.

Example

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$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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- Here we will take a different approach.

The powers of A

- By direct computation,

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- By induction, we get $A^{2m+1} = 4^m A, A^{2m+2} = 4^m A^2$ for all $m \geq 0$.

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- By induction, we get $A^{2m+1} = 4^m A, A^{2m+2} = 4^m A^2$ for all $m \geq 0$.
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- END OF LECTURE 22