

LINEAR ALGEBRA -II

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Lecture 22: Path counting

- ▶ Recall: We recall some definitions and the spectral theorem for normal matrices.

Self-adjoint and normal matrices

- **Definition 20.1:** (i) A complex square matrix A is said to be **self-adjoint** if $A^* = A$. (ii) A complex square matrix A is said to be **normal** if $A^*A = AA^*$.

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- ▶ In particular, every real symmetric matrix is self-adjoint.
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- ▶ Note that diagonal entry of every self-adjoint matrix is real as $\overline{a_{ii}} = a_{ii}$ for every i .

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- ▶ Example 20.3: Consider

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then C is not normal.

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- ▶ Computing the first diagonal entries of T^*T and TT^* , as T is normal, we get

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- ▶ In other words, T is diagonal. ■

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- ▶ **Proposition 20.6:** Suppose B is unitarily equivalent to A . Then B is normal (resp. self-adjoint, unitary, projection) if and only if A is normal (resp. self-adjoint, unitary, projection).
- ▶ **Proof:** Suppose U is a unitary such that $B = UAU^*$. Then $B^*B = (UAU^*)^*(UAU^*) = UA^*UU^*AU = UA^*AU^*$. Similarly, $BB^* = UAA^*U^*$. Now the result follows easily.

Spectral theorem (Version -I)

- **Theorem 20.7 (Spectral Theorem-I):** Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that

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- ▶ **Proof:** Suppose A is a normal matrix.
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- ▶ Note that A and T are unitarily equivalent. Consequently T is normal. Then by Theorem 20.4, as T is both upper triangular and normal it must be diagonal. Taking $D = T$, we have $A = UDU^*$ and we are done.

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- ▶ Since every diagonal matrix is normal, D is normal.
- ▶ Then as A is unitarily equivalent to D , A is also normal. ■.

Consequences of the spectral theorem

- **Corollary 21.1:** Let A be an $n \times n$ complex matrix. Then A is normal if and only if there exists an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of \mathbb{C}^n consisting of eigenvectors of A .

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- ▶ Take $U = [v_1, \dots, v_n]$. Then U is a unitary and $AU = UD$. Hence $A = UDU^*$. Consequently A is normal. ■

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- ▶ Hence the eigenvalues of A are 1 and 3.
- ▶ Solving corresponding eigen equations we see that

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Continuation

- ▶ Normalizing these eigenvectors, and taking them as columns we get a unitary,

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- ▶ Alternatively,

$$A = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} U^*.$$

Terminology and notation

- **Definition 21.3:** Let A be a complex square matrix. Then

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- Note that for a matrix A , if a_1, a_2, \dots, a_n are eigenvalues of A , then

$$\sigma(A) = \{a_1, a_2, \dots, a_n\}$$

Some characterizations

- **Theorem 21.4:** Let A be a normal matrix. Then,
- (i) A is self-adjoint iff $\sigma(A) \subset \mathbb{R}$.
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- ▶ If A is self-adjoint, then $A = A^*$. Hence,

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- ▶ Multiplication by U^* , U , yields $D = D^*$. Since D is diagonal, this means that all the diagonal entries are real. Hence

$$\sigma(A) \subset \mathbb{R}.$$

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- ▶ Similarly (ii) and (iii) of this theorem do not hold without the assumption of normality.

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- ▶ Conversely, suppose $\|Ax\| = \|A^*x\|$, $\forall x \in \mathbb{C}^n$.
- ▶ Then $\langle x, A^*Ax \rangle = \langle x, AA^*x \rangle$, $\forall x \in \mathbb{C}^n$.
- ▶ Polarization identity yields,

$$\langle x, A^*Ay \rangle = \langle x, AA^*y \rangle, \quad \forall x, y \in \mathbb{C}^n.$$

Another characterization of a normal matrix

- ▶ **Theorem 21.6:** Let A be an $n \times n$ complex matrix. Then A is normal iff $\|Ax\| = \|A^*x\|$ for all $x \in \mathbb{C}^n$.
- ▶ **Proof:** Suppose A is normal. Then for $x \in \mathbb{C}^n$,

$$\begin{aligned}\|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= \langle x, A^*Ax \rangle \\ &= \langle x, AA^*x \rangle \\ &= \langle A^*x, A^*x \rangle \\ &= \|A^*x\|^2.\end{aligned}$$

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- ▶ Hence $A^*A = AA^*$. ■

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- ▶ In particular, $\|(A - cI)x\| = 0$ if and only if $\|(A - cI)^*x\| = 0$.
- ▶ As $(A - cI)^* = A^* - \bar{c}I$ this shows that x is an eigenvector for A with eigenvalue c if and only if it is an eigenvector for A^* with eigenvalue \bar{c} .

- ▶ **Exercise 22.1:** Show that eigenvectors of a normal matrix corresponding to distinct eigenvalues are mutually orthogonal.

Exercises

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- ▶ **Exercise 22.3:** Suppose A is a normal matrix. Show that there exists a polynomial p (depending upon A) such that $A^* = p(A)$. Show that this result may not hold without the assumption that A is normal.

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- ▶ **Exercise 22.4:** Suppose A is a normal matrix. Then show that a matrix B commutes with A if and only if it commutes with A^* .

Path counting

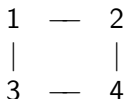
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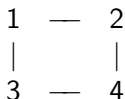
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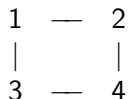
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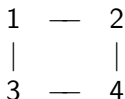
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- ▶ Examples of such paths:
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Continuation

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- ▶ What is the total number of such paths?
- ▶ What is the number of paths from 1 to 4 in 100 steps?
- ▶ How do we count such paths?

Continuation

- ▶ A graph G is a pair (V, E) where V is a set called the set of vertices and E is a collection of pairs of the form $\{x, y\}$ where $x, y \in V$, with $x \neq y$, called the edges of the graph.

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- ▶ We are talking about simple graphs, which means no multiple edges between vertices and no loops and the edges are undirected. There are more general notions of graphs which we will not go into.
- ▶ For the square graph example above: $V = \{1, 2, 3, 4\}$ and

$$E = \{\{1, 2\}, \{2, 4\}, \{1, 3\}, \{3, 4\}\}.$$

- ▶ In the following, we take $V = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ and so E is a collection of pairs of the form $\{i, j\}$ with $i, j \in V$.

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- ▶ For $m \geq 2$, a path of length m from i to j , is a tuple of the form $(i, k_1, k_2, \dots, k_{m-1}, j)$ where $\{i, k_1\}, \{k_1, k_2\}, \dots, \{k_{m-1}, j\}$ are edges, i.e., they are elements of E .

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- ▶ Note that k_j 's need not be distinct.

Adjacency matrix

- **Definition 22.6:** The **adjacency** matrix $A = [a_{ij}]_{1 \leq i, j \leq n}$ of a graph $G = (V, E)$ where $V = \{1, 2, \dots, n\}$ is defined by taking

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

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$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

- Note that A is a real symmetric matrix and the diagonal entries are all equal to zero.

Two step paths

► We have

$$\begin{aligned}(A^2)_{ij} &= \sum_{k=1}^n a_{ik} a_{kj} \\&= \sum_{k: a_{ik} \cdot a_{kj} \neq 0} a_{ik} a_{kj} \\&= \#\{k : a_{ik} \cdot a_{kj} \neq 0\} \\&= \#\{k : a_{ik} \neq 0, \ a_{kj} \neq 0\} \\&= \#\{k : (i, k), (k, j) \in E\} \\&= \text{The number of two step paths from } i \text{ to } j.\end{aligned}$$

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- ▶ Therefore, $(i, j)^{\text{th}}$ -entry of A^2 is the number of two step paths from (i, j) .

m -step paths

- ▶ For $m \geq 2$,

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$$\begin{aligned} & (A^m)_{ij} \\ = & \sum_{k_1, k_2, \dots, k_{m-1}} a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \\ = & \#\{(k_1, k_2, \dots, k_{m-1}) : a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{m-2} k_{m-1}} a_{k_{m-1} j} \neq 0\} \\ = & \#\{(i, k_1, \dots, k_{m-1}, j) : (i, k_1), (k_1, k_2), \dots, (k_{m-1}, j) \in E\} \\ = & \text{Number of paths of length } m \text{ from } i \text{ to } j. \end{aligned}$$

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- ▶ In other words, $(i, j)^{\text{th}}$ -entry of A^m is exactly the number of paths of length m from i to j in the graph G , where A is the adjacency matrix of the graph G .

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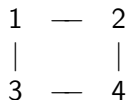
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- ▶ Consequently $A^m = UD^mU^*$. This allows us to compute the number of paths of length m between any two vertices.

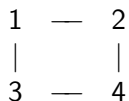
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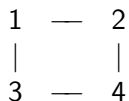


- ▶ The adjacency matrix of this graph is given by

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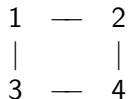
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- ▶ You may compute the eigenvalues of A and diagonalize it to compute the powers.
- ▶ Here we will take a different approach.

The powers of A

- By direct computation,

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- By induction, we get $A^{2m+1} = 4^m A$, $A^{2m+2} = 4^m A^2$ for all $m \geq 0$.

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- **END OF LECTURE 22**