LINEAR ALGEBRA -II

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Lecture 23: Spectral theorem -II and III

- Reference: Graphs and Matrices by R. Bapat

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- For further information on connections between graph theory and matrix theory.

Self-adjoint and normal matrices

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- Note that diagonal entry of every self-adjoint matrix is real as $\overline{a_{ii}} = a_{ii}$ for every i.

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- Now we will present this Theorem in a different way.
- Consider the set up as above. Let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A.
- Recall that the diagonal entries of D are the eigenvalues of A, as the characteristic polynomial of A and D are same.

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- Suppose that a_j appears n_j times in the diagonal with $1 \leq n_j$ and $n_1 + n_2 + \cdots + n_k = n$.
- Without loss of generality, we may assume that repeated entries are clubbed together, that is, the diagonal entries of D are equal to

$$(a_1, a_1, \ldots, a_1, a_2, a_2, \ldots a_2, a_3, a_3, \ldots, a_k, a_k)$$

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  where a_j appears n_j times.
- If I_{n_j} denotes the identity matrix of size $n_j \times n_j$, the matrix D can be written as:

$$D = \begin{bmatrix} a_1 I_{n_1} & 0 & 0 & \ldots & 0 \\ 0 & a_2 I_{n_2} & 0 & \ldots & 0 \\ 0 & 0 & a_3 I_{n_3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_k I_{n_k} \end{bmatrix}$$

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- Clearly $Q_1, Q_2, \dots, Q_k$ are projections, $Q_i Q_j = 0, \quad \text{for } i \neq j$ (they are mutually orthogonal) and $Q_1 + Q_2 + \cdots + Q_k = I$.

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- From $P_j = UQ_jU^*, 1 \le j \le k$, it is clear that $P_1, P_2, \ldots, P_k$ are projections such that $P_iP_j = 0$ for $i \neq j$ and

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- Theorem 23.1 (Spectral Theorem -II): Let A be a normal matrix and let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A. Then there exist mutually orthogonal projections $P_1, P_2, \ldots, P_k$, such that

$$\begin{aligned} I &= P_1 + P_2 + \cdots + P_k; \\ A &= a_1 P_1 + a_2 P_2 + \cdots + a_k P_k. \end{aligned}$$

Orthogonal Direct sums

- Definition 23.2: Suppose $M_1, M_2, \ldots, M_k$ are mutually orthogonal subspaces of a finite dimensional inner product space V such that every vector x in V decomposes uniquely as

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- Note that:

$$\langle y_1 \oplus y_2 \oplus \cdots \oplus y_k, z_1 \oplus z_2 \oplus \cdots \oplus z_k \rangle = \sum_{j=1}^{k} \langle y_j, z_j \rangle.$$

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- Now in Spectral theorem-II, taking $M_j = P(\mathbb{C}^n) = \{P_j x : x \in \mathbb{C}^n\}$, we see that $\mathbb{C}^n$ is a direct sum of $M_1, M_2, \ldots, M_k$.

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- That is, every vector x in $\mathbb{C}^n$ decomposes uniquely as $x = (P_1 + P_2 + \cdots + P_k)x = P_1x + P_2x + \cdots + P_kx$ with $P_jx \in M_j$.

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$$Ax = (a_1 P_1 + a_2 P_2 + \cdots + a_k P_k)x = a_1 P_1 x + a_2 P_2 x + \cdots + a_k P_k x$$

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- In other words, M_j is the eigenspace of A with respect to eigenvalue a_j.

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- In other words, we have proved the following statement:
- Theorem 23.4 (Spectral theorem -III): Let A be a normal matrix. Then the eigenspaces of distinct eigenvalues of A are mutually orthogonal and $\mathbb{C}^n$ is their direct sum.
- Clearly given the normal matrix A, the decomposition of $\mathbb{C}^n$ as in this theorem is uniquely determined and so the corresponding projections are also uniquely determined. This also shows that the decomposition of A as in Spectral Theorem -II:

$$A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k, I = P_1 + P_2 + \cdots + P_k$$

where $P_1, P_2, \ldots, P_k$ are mutually orthogonal projections is unique up to permutation.

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