LINEAR ALGEBRA -II

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Lecture 24: Polynomial spectral mapping theorem

- We recall different versions of the spectral theorem.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that
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- Now we will present this Theorem in a different way.
- Consider the set up as above. Let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A.
- Recall that the diagonal entries of D are the eigenvalues of A, as the characteristic polynomial of A and D are same.

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- Suppose that a_j appears n_j times in the diagonal with $1 \leq n_j$ and $n_1 + n_2 + \cdots + n_k = n$.
- Without loss of generality, we may assume that repeated entries are clubbed together, that is, the diagonal entries of D are equal to

$$(a_1, a_1, \ldots, a_1, a_2, a_2, \ldots a_2, a_3, a_3, \ldots, a_k, a_k)$$

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  where a_j appears n_j times.
- If I_{n_j} denotes the identity matrix of size $n_j \times n_j$, the matrix D can be written as:

$$D = \begin{bmatrix} a_1 I_{n_1} & 0 & 0 & \ldots & 0 \\ 0 & a_2 I_{n_2} & 0 & \ldots & 0 \\ 0 & 0 & a_3 I_{n_3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_k I_{n_k} \end{bmatrix}$$

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- Clearly $Q_1, Q_2, \dots, Q_k$ are projections, $Q_i Q_j = 0, \quad \text{for } i \neq j$ (they are mutually orthogonal) and $Q_1 + Q_2 + \cdots + Q_k = I$.

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- Then we have

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- From $P_j = UQ_jU^*, 1 \leq j \leq k$, it is clear that $P_1, P_2, \ldots, P_k$ are projections such that $P_iP_j = 0$ for $i \neq j$ and

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- Theorem 23.1 (Spectral Theorem -II): Let A be a normal matrix and let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A. Then there exist mutually orthogonal projections $P_1, P_2, \ldots, P_k$, such that

$$\begin{aligned} I &= P_1 + P_2 + \cdots + P_k; \\ A &= a_1 P_1 + a_2 P_2 + \cdots + a_k P_k. \end{aligned}$$

Orthogonal Direct sums

- Definition 23.2: Suppose $M_1, M_2, \ldots, M_k$ are mutually orthogonal subspaces of a finite dimensional inner product space V such that every vector x in V decomposes uniquely as

$$x = y_1 + y_2 + \cdots + y_k$$

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- (Notation) Sometimes this is denoted by

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- Note that:

$$\langle y_1 \oplus y_2 \oplus \cdots \oplus y_k, z_1 \oplus z_2 \oplus \cdots \oplus z_k \rangle = \sum_{j=1}^{k} \langle y_j, z_j \rangle.$$

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- Now in Spectral theorem-II, taking $M_j = P(\mathbb{C}^n) = \{P_j x : x \in \mathbb{C}^n\}$, we see that $\mathbb{C}^n$ is a direct sum of $M_1, M_2, \ldots, M_k$.

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- That is, every vector x in $\mathbb{C}^n$ decomposes uniquely as $x = (P_1 + P_2 + \cdots + P_k)x = P_1 x + P_2 x + \cdots + P_k x$ with $P_j x \in M_j$.

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- We also note that

$$Ax = (a_1 P_1 + a_2 P_2 + \cdots + a_k P_k)x = a_1 P_1 x + a_2 P_2 x + \cdots + a_k P_k x$$

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- Exercise 23.3: Show that

$$M_j = \{y \in \mathbb{C}^n : Ay = a_jy\}.$$

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- In other words, M_j is the eigenspace of A with respect to eigenvalue a_j.

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- Theorem 23.4 (Spectral theorem -III): Let A be a normal matrix. Then the eigenspaces of distinct eigenvalues of A are mutually orthogonal and $\mathbb{C}^n$ is their direct sum.

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- In other words, we have proved the following statement:
- Theorem 23.4 (Spectral theorem -III): Let A be a normal matrix. Then the eigenspaces of distinct eigenvalues of A are mutually orthogonal and $\mathbb{C}^n$ is their direct sum.
- Clearly given the normal matrix A, the decomposition of $\mathbb{C}^n$ as in this theorem is uniquely determined and so the corresponding projections are also uniquely determined. This also shows that the decomposition of A as in Spectral Theorem -II:

$$A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k, I = P_1 + P_2 + \cdots + P_k$$

where $P_1, P_2, \ldots, P_k$ are mutually orthogonal projections is unique up to permutation.

Non-uniqueness in diagonalization

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- Is U unique up to multiplication by scalar when D is fixed?
- Ans: No. If $A = I$, then $A = UIU^*$ for any unitary U. Hence U is not unique even up to scalar.

Polynomial spectral mapping theorem

- Theorem 24.1: Let A be an $n \times n$ matrix and let $d_1, d_2, \ldots, d_n$ be the eigenvalues of A. Then for any complex polynomial q, the eigenvalues of $q(A)$ are $q(d_1), q(d_2), \ldots, q(d_n)$.

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- Now it is easy to see that the diagonal entries of T^2 are $d_1^2, d_2^2, \dots, d_n^2$.
- More generally, for any $k \in \mathbb{N}$ the diagonal entries of T^k are $d_1^k, d_2^k, \dots, d_n^k$.

Continuation

- Now suppose $q(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$, then the diagonal entries of $q(T)$ are $q(d_1), q(d_2), \ldots, q(d_n)$.

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- We have
$$q(A) = Uq(T)U^*$$
and $q(T)$ is upper triangular, Therefore $q(A)$ and $q(T)$ have same characteristic polynomial and hence same set of eigenvalues.

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- This proves the theorem.
- Exercise 24.2: Find an alternative proof which does not use upper triangularization.

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- and $I = UIU^*$. ■

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Polynomials of normal matrices -II

- Theorem 24.4: Let A be a normal matrix. Suppose

$$A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k,$$

is the spectral decomposition of A (This means that $a_1, a_2, \ldots, a_k$ are distinct eigenvalues of A and $P_1, P_2, \ldots, P_k$ are mutually orthogonal projections such that $P_1 + P_2 + \cdots + P_k = I$.)

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- Then for any complex polynomial q,

$$q(A) = q(a_1)P_1 + q(a_2)P_2 + \cdots + q(a_k)P_k.$$

- Proof: We have

$$\begin{aligned} A^2 &= (a_1 P_1 + a_2 P_2 + \cdots + a_k P_k)(a_1 P_1 + a_2 P_2 + \cdots + a_k P_k) \\ &= a_1^2 P_1 + a_2^2 P_2 + \cdots + a_k^2 P_k \end{aligned}$$

as $P_i P_j = \delta_{ij} P_j$.

Continuation

- By induction,

$$A^m = a_1^m P_1 + a_2^m P_2 + \cdots + a_k^m P_k$$

for all $m \geq 1$ and for $m = 0$, $A^0 = I = P_1 + P_2 + \cdots + P_k$.

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- Now the result follows by taking linear combinations of the powers of A. ■
- Remark 24.5: It is to be noted that

$$q(A) = q(a_1)P_1 + q(a_2)P_2 + \cdots + q(a_k)P_k.$$

may not be the spectral decomposition of $q(A)$ as $q(a_1), \ldots, q(a_k)$ may not be distinct.

Functional Calculus

- The last two theorems suggest that for a normal matrix A, if f is a function defined on $\sigma(A)$ (the spectrum of A) we may define $f(A)$ by taking

$$
\begin{aligned}
f(A) &:= U \begin{bmatrix} f(d_1) & 0 & \dots & 0 \\ 0 & f(d_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(d_n) \end{bmatrix} U^* \\
&= f(a_1)P_1 + f(a_2)P_2 + \cdots + f(a_k)P_k.
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- For instance we can define $\sin(A)$, $\cos(A)$, e^A etc by this method.
- At the moment this is only a definition. But it has many natural properties. Studying this concept not only for matrices but also for operators (infinite dimensional matrices) is the subject of Functional Calculus.

Examples

- Recall:
- Example 21.2 and 24.6: Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

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- Hence the eigenvalues of A are 1 and 3.
- Solving corresponding eigen equations we see that

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Continuation

- Normalizing these eigenvectors, and taking them as columns we get a unitary,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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- Alternatively,

$$A = U \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} U^*.$$

Continuation

- We have the spectral decomposition of A as

$$A = 3P_1 + 1.P_2$$

where

$$P_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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- Note that P_1, P_2 are mutually orthogonal and $P_1 + P_2 = I$.

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- Compute the spectral decomposition of U.
- Challenge Question: Every time you diagonalize you get a unitary. Continue diagonalizing these unitaries. Does the process terminate or does it become cyclic?

Another example

- Example 24.8: Let B be the $n \times n$ matrix defined by

$$b_{ij} = \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{otherwise.} \end{cases}$$

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- Draw K_1, K_2, K_3, K_4

Continuation

- We write B as $B = nP - I$ where P is the matrix with
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- Take $Q = (I - P)$.
- Then $B = nP - (P + Q) = (n-1)P + (-1)Q$

Continuation

- Observe that P, Q are mutually orthogonal projections whose sum is I. It follows that

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- END OF LECTURE 24.