LINEAR ALGEBRA -II

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Lecture 25: Hadamard matrices and circulant matrices

- We recall different versions of the spectral theorem.

Spectral theorem (Version -I)

- Theorem 20.7 (Spectral Theorem-I): Let A be a complex matrix. Then there exists a unitary matrix U and a diagonal matrix D such that

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- Now we will present this Theorem in a different way.
- Consider the set up as above. Let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A.
- Recall that the diagonal entries of D are the eigenvalues of A, as the characteristic polynomial of A and D are same.

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- Suppose that a_j appears n_j times in the diagonal with $1 \leq n_j$ and $n_1 + n_2 + \cdots + n_k = n$.
- Without loss of generality, we may assume that repeated entries are clubbed together, that is, the diagonal entries of D are equal to

$$(a_1, a_1, \ldots, a_1, a_2, a_2, \ldots a_2, a_3, a_3, \ldots, a_k, a_k)$$

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  where a_j appears n_j times.
- If I_{n_j} denotes the identity matrix of size $n_j \times n_j$, the matrix D can be written as:

$$D = \begin{bmatrix} a_1 I_{n_1} & 0 & 0 & \ldots & 0 \\ 0 & a_2 I_{n_2} & 0 & \ldots & 0 \\ 0 & 0 & a_3 I_{n_3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_k I_{n_k} \end{bmatrix}$$

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- Clearly $Q_1, Q_2, \dots, Q_k$ are projections, $Q_i Q_j = 0, \quad \text{for } i \neq j$ (they are mutually orthogonal) and $Q_1 + Q_2 + \cdots + Q_k = I$.

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A &= UDU^* \\
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&= a_1UQ_1U^* + a_2UQ_2U^* + \cdots + a_kUQ_kU^* \\
&= a_1P_1 + a_2P_2 + \cdots + a_kP_k.
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- From $P_j = UQ_jU^*, 1 \leq j \leq k$, it is clear that $P_1, P_2, \ldots, P_k$ are projections such that $P_iP_j = 0$ for $i \neq j$ and

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- Theorem 23.1 (Spectral Theorem -II): Let A be a normal matrix and let $a_1, a_2, \ldots, a_k$ be the distinct eigenvalues of A. Then there exist mutually orthogonal projections $P_1, P_2, \ldots, P_k$, such that

$$\begin{aligned} I &= P_1 + P_2 + \cdots + P_k; \\ A &= a_1 P_1 + a_2 P_2 + \cdots + a_k P_k. \end{aligned}$$

Orthogonal Direct sums

- Definition 23.2: Suppose $M_1, M_2, \ldots, M_k$ are mutually orthogonal subspaces of a finite dimensional inner product space V such that every vector x in V decomposes uniquely as

$$x = y_1 + y_2 + \cdots + y_k$$

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- (Notation) Sometimes this is denoted by

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- Note that:

$$\langle y_1 \oplus y_2 \oplus \cdots \oplus y_k, z_1 \oplus z_2 \oplus \cdots \oplus z_k \rangle = \sum_{j=1}^{k} \langle y_j, z_j \rangle.$$

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- Now in Spectral theorem-II, taking $M_j = P(\mathbb{C}^n) = \{P_j x : x \in \mathbb{C}^n\}$, we see that $\mathbb{C}^n$ is a direct sum of $M_1, M_2, \ldots, M_k$.

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- That is, every vector x in $\mathbb{C}^n$ decomposes uniquely as $x = (P_1 + P_2 + \cdots + P_k)x = P_1x + P_2x + \cdots + P_kx$ with $P_jx \in M_j$.

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- We also note that

$$Ax = (a_1P_1 + a_2P_2 + \cdots + a_kP_k)x = a_1P_1x + a_2P_2x + \cdots + a_kP_kx$$

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- Exercise 23.3: Show that

  $$M_j = \{y \in \mathbb{C}^n : Ay = a_j y\}.$$

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- In other words, M_j is the eigenspace of A with respect to eigenvalue a_j.

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- Theorem 23.4 (Spectral theorem -III): Let A be a normal matrix. Then the eigenspaces of distinct eigenvalues of A are mutually orthogonal and $\mathbb{C}^n$ is their direct sum.

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- In other words, we have proved the following statement:
- Theorem 23.4 (Spectral theorem -III): Let A be a normal matrix. Then the eigenspaces of distinct eigenvalues of A are mutually orthogonal and $\mathbb{C}^n$ is their direct sum.
- Clearly given the normal matrix A, the decomposition of $\mathbb{C}^n$ as in this theorem is uniquely determined and so the corresponding projections are also uniquely determined. This also shows that the decomposition of A as in Spectral Theorem -II:

$$A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k, I = P_1 + P_2 + \cdots + P_k$$

where $P_1, P_2, \ldots, P_k$ are mutually orthogonal projections is unique up to permutation.

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- Is U unique up to multiplication by scalar when D is fixed?
- Ans: No. If $A = I$, then $A = UIU^*$ for any unitary U. Hence U is not unique even up to scalar.

Polynomial spectral mapping theorem

- Theorem 24.1: Let A be an $n \times n$ matrix and let $d_1, d_2, \ldots, d_n$ be the eigenvalues of A. Then for any complex polynomial q, the eigenvalues of $q(A)$ are $q(d_1), q(d_2), \ldots, q(d_n)$.

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- Now it is easy to see that the diagonal entries of T^2 are $d_1^2, d_2^2, \ldots, d_n^2$.
- More generally, for any $k \in \mathbb{N}$ the diagonal entries of T^k are $d_1^k, d_2^k, \ldots, d_n^k$.

Continuation

- Now suppose $q(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$, then the diagonal entries of $q(T)$ are $q(d_1), q(d_2), \ldots, q(d_n)$.

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- We have
$$q(A) = Uq(T)U^*$$
and $q(T)$ is upper triangular, Therefore $q(A)$ and $q(T)$ have same characteristic polynomial and hence same set of eigenvalues.

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- Exercise 24.2: Find an alternative proof which does not use upper triangularization.

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Polynomials of normal matrices -II

- Theorem 24.4: Let A be a normal matrix. Suppose

$$A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k,$$

is the spectral decomposition of A (This means that $a_1, a_2, \ldots, a_k$ are distinct eigenvalues of A and $P_1, P_2, \ldots, P_k$ are mutually orthogonal projections such that $P_1 + P_2 + \cdots + P_k = I$.)

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- Then for any complex polynomial q,

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- Proof: We have

$$\begin{aligned} A^2 &= (a_1 P_1 + a_2 P_2 + \cdots + a_k P_k)(a_1 P_1 + a_2 P_2 + \cdots + a_k P_k) \\ &= a_1^2 P_1 + a_2^2 P_2 + \cdots + a_k^2 P_k \end{aligned}$$

as $P_i P_j = \delta_{ij} P_j$.

Continuation

- By induction,

$$A^m = a_1^m P_1 + a_2^m P_2 + \cdots + a_k^m P_k$$

for all $m \geq 1$ and for $m = 0$, $A^0 = I = P_1 + P_2 + \cdots + P_k$.

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- Now the result follows by taking linear combinations of the powers of A. ■
- Remark 24.5: It is to be noted that

$$q(A) = q(a_1)P_1 + q(a_2)P_2 + \cdots + q(a_k)P_k.$$

  may not be the spectral decomposition of $q(A)$ as $q(a_1), \ldots, q(a_k)$ may not be distinct.

Functional Calculus

- The last two theorems suggest that for a normal matrix A, if f is a function defined on $\sigma(A)$ (the spectrum of A) we may define $f(A)$ by taking

$$\begin{aligned} f(A) &:= U \begin{bmatrix} f(d_1) & 0 & \dots & 0 \\ 0 & f(d_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(d_n) \end{bmatrix} U^* \\ &= f(a_1)P_1 + f(a_2)P_2 + \cdots + f(a_k)P_k. \end{aligned}$$

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- For instance we can define $\sin(A)$, $\cos(A)$, e^A etc by this method.
- At the moment this is only a definition. But it has many natural properties. Studying this concept not only for matrices but also for operators (infinite dimensional matrices) is the subject of Functional Calculus.

Hadamard matrices

- Definition 25.1: A square matrix is said to be a Hadamard matrix if every entry of it is ± 1 and its rows are mutually orthogonal.

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are Hadamard matrices.

- Note that if an $n \times n$ matrix H is a Hadamard matrix, clearly

$$HH^t = nI.$$

Therefore $\frac{1}{\sqrt{n}}H^t$ is the inverse of $\frac{1}{\sqrt{n}}H$. Alternatively, $\frac{1}{\sqrt{n}}H$ is an orthogonal matrix. Consequently we also have $H^tH = nI$. Therefore columns of H are also mutually orthogonal.

A construction

- Proposition 25.3: If H is an $n \times n$ Hadamard matrix then the block matrix
$$K = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$
is a $2n \times 2n$ Hadamard matrix.

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- Proof By block matrix multiplication,

$$\begin{aligned} KK^t &= \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \cdot \begin{bmatrix} H^t & H^t \\ H^t & -H^t \end{bmatrix} \\ &= \begin{bmatrix} nI + nI & nI - nI \\ nI - nI & nI + nI \end{bmatrix} \\ &= 2n \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

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- This proves the claim. ■

Continuation

- Applying this construction to H_2 defined above we get a 4×4 Hadamard matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

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- By induction, we can construct a Hadamard matrix of order 2^n for every $n \geq 0$.

Multiples of 4

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- If H is a Hadamard matrix, and if we replace a row with its negative, clearly it stays as Hadamard matrix.
- Therefore we may assume that the first entry of every row is $+1$. (That is $h_{j1} = +1, \forall j$.)
- In other words now the first column has only '$+1$'s. This forces that every other column has equal number of positive and negative entries. In particular n must be even.

Continuation

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- If H is a Hadamard matrix and if we permute the rows it would stay as a Hadamard matrix. So we may assume that first k entries of second column are positive and the next k are negative. That is:

$$h_{j2} = \begin{cases} +1 & \text{if } 1 \leq j \leq k; \\ -1 & \text{if } (k+1) \leq j \leq 2k. \end{cases}$$

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- Since $n > 2$, and n is even we can consider third and fourth columns of H.

Continuation

- Suppose among the first k entries of the third column r entries are positive and $(k - r)$ are negative and among the remaining k entries s are positive and $(k - s)$ are negative.

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- Since the total number of positive entries in a column has to be k, we get $r + s = k$.

Continuation

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- Counting the positions where both the entry of second and third column have same sign we get $r + (k - s) = k$ or equivalently $r = s$.

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- Counting the positions where both the entry of second and third column have same sign we get $r + (k - s) = k$ or equivalently $r = s$.
- Then $k = r + s = 2r$ is even.
- Consequently $n = 2k$ is a multiple of 4.

Hadamard's Conjecture

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- The conjecture is still open.
- As per Wikipedia currently 668 is the smallest number for which we don't know the existence of a Hadamard matrix.

Complex Hadamard matrices

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$$AA^* = A^*A = nI.$$

- In the following Example we find it convenient to index the rows and columns of the matrix from 0 to $(n-1)$ instead of 1 to n.
- Example 25.6: For $n \geq 1$, consider the matrix $W = [w_{jk}]_{0 \leq j,k \leq (n-1)}$ defined by

$$w_{jk} = e^{\frac{2\pi ijk}{n}}.$$

Then W is a complex Hadamard matrix (Prove it.)

Finite Fourier transform

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- It has several practical applications.
- For instance, for $n = 3$ we have

$$W = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$.

Circulant matrices

- Definition 25.6: Fo $n \geq 2$, an $n \times n$ matrix C is said to be a circulant matrix if

$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix}$$

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  for some $c_0, c_1, \dots, c_{n-1} \in \mathbb{C}$.
- Suppose A is as above. Consider the matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Continuation

- Then it is easily seen that

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- Note that S is a permutation matrix and in particular it is a unitary.

Continuation

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- Taking $\omega = e^{\frac{2\pi i}{n}}$. $\sigma(S) = \{1, \omega, \omega^2, \omega^{n-1}\}$.

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- Let D be the diagonal matrix with diagonal entries $1, \omega, \ldots, w^{n-1}$.
- So we have $d_{jk} = \delta_{jk}\omega^j, 0 \leq j, k \leq (n-1)$.

Continuation

- Note that on indexing the rows and columns of S from 0 to $(n-1)$, we have

$$s_{kl} = \begin{cases} 1 & \text{if } k = (l+1) \\ 0 & \text{otherwise} \end{cases}$$

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- As $W^*W = nI$, $\frac{1}{\sqrt{n}}$ is a unitary. Therefore $\frac{1}{\sqrt{n}}WS = D.\frac{1}{\sqrt{n}}W$, or $S = \frac{1}{n}W^*DW$ is the diagonalization of S.

Continuation

- Recall that the Circulant matrix

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- END OF LECTURE 25.