LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 26: Positive matrices

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- Warning: A positive matrix need not have positive entries. It can have negative entries and also complex entries.
- Matrices whose entries are positive would be called as entrywise positive matrices. That is also an important class, but we will not be studying them now.

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- (v) $A = A^*$ and eigenvalues of A are non-negative.
- (vi) $A = S^2$ for some self-adjoint $n \times n$ matrix S.

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- (iii) $\Rightarrow$ (iv). We have $a_{ij} = \langle v_i, v_j \rangle, \quad \forall i, j$.
- Now for any $x \in \mathbb{C}^n$:

$$\begin{aligned}
\langle x, Ax \rangle &= \sum_{i=1}^{n} \overline{x_i}(Ax)_i \\
&= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} a_{ij} x_j \\
&= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} \langle v_i, v_j \rangle . x_j
\end{aligned}$$

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- Therefore,

$$\begin{aligned}\langle x, Ax\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x_i v_i, x_j v_j\rangle \\ &= \langle\sum_{i=1}^{n} x_i v_i, \sum_{j=1}^{n} x_j v_j\rangle \\ &= \langle y, y\rangle\end{aligned}$$

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$$\langle x, Ax\rangle \geq 0.$$

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- First we want to show that $A = A^*$. Here we use the polarization identity and the fact that if $\langle v, w\rangle$ is real then $\langle v, w\rangle = \langle w, v\rangle$. For all x, y,

$$\begin{aligned}
\langle x, Ay\rangle &= \frac{1}{4}\sum_{j=0}^{3} i^{-j}\langle (x + i^j y), A(x + i^j y)\rangle \\
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- implies that $a \geq 0$ as $\langle x, x\rangle \neq 0$.

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- Take
$$S = U \begin{bmatrix} \sqrt{d_1} & 0 & \ldots & 0 \\ 0 & \sqrt{d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{d_n} \end{bmatrix} U^*.$$

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- Then clearly S is self-adjoint and $A = S^2$.

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- Find all self-adjoint operators S such that $R = S^2$. (Exercise)

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