LINEAR ALGEBRA -II

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Lecture 27: Sums, products and square roots of positive matrices

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- Warning: A positive matrix need not have positive entries. It can have negative entries and also complex entries.
- Matrices whose entries are positive would be called as entrywise positive matrices. That is also an important class, but we will not be studying them now.

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- (v) $A = A^*$ and eigenvalues of A are non-negative.
- (vi) $A = S^2$ for some self-adjoint $n \times n$ matrix S.

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- (iii) $\Rightarrow$ (iv). We have $a_{ij} = \langle v_i, v_j \rangle, \quad \forall i, j$.
- Now for any $x \in \mathbb{C}^n$:

$$\begin{aligned}
\langle x, Ax \rangle &= \sum_{i=1}^{n} \overline{x_i}(Ax)_i \\
&= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} a_{ij} x_j \\
&= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} \langle v_i, v_j \rangle . x_j
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- Therefore,

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$$\begin{aligned}
\langle x, Ay \rangle &= \frac{1}{4} \sum_{j=0}^{3} i^{-j} \langle (x + i^j y), A(x + i^j y) \rangle \\
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- implies that $a \geq 0$ as $\langle x, x\rangle \neq 0$.

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- Take
$$S = U \begin{bmatrix} \sqrt{d_1} & 0 & \ldots & 0 \\ 0 & \sqrt{d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{d_n} \end{bmatrix} U^*.$$

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- Then clearly S is self-adjoint and $A = S^2$.

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- Clearly R is self-adjoint. We have the characteristic polynomial of R, as

$$p(x) = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-1)(x-3).$$

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- Find all self-adjoint operators S such that $R = S^2$. (Exercise)

Cartesian decomposition

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- This proves uniqueness. ■
- This is known as Cartesian decomposition.

Constructing positive matrices out of positive matrices

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- Note that this theorem does not follow directly from the definition of positivity or from the eigenvalue criterion.
- This theorem shows that the set of $n \times n$ positive matrices has 'cone' structure: It is closed under taking sums and it is closed under multiplication by positive scalar.

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- Proof: As A is positive, $A = D^*D$ for some matrix D.
- Now, $B^*AB = B^*D^*DB = (DB)^*(DB)$. Hence B^*AB is positive from the definition of positivity. We may also see this from looking at the quadratic form. ■

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- Proof: The first part is clear as the trace and determinant of a matrix are respectively the sum and the product of its eigenvalues and a positive matrix has non-negative eigenvalues. The second claim follows from $a_{ii} = \langle v_i, v_i \rangle$ in part (iv) of the characterization.

Gram matrices

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$$G = \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{bmatrix}.$$
- We know that G is positive. Hence its determinant is positive. So we get $\langle x, x \rangle . \langle y, y \rangle - \langle x, y \rangle . \langle y, x \rangle \geq 0$.
- In other words, we have the Cauchy-Schwarz inequality:
$$|\langle x, y \rangle|^2 \leq \|x\|^2 . \|y\|^2.$$

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- Now suppose B is positive and $A = B^2$.
- Let $b_1, b_2, \ldots, b_k$ the distinct eigenvalues of B and $B = b_1 Q_1 + b_2 Q_2 + \cdots + b_k Q_k$ be the spectral decomposition of B.

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- In other words, if $A = a_1 P_1 + a_2 P_2 + \cdots + a_k P_k$ is the spectral decomposition of A, then $B = \sqrt{a_1} P_1 + \sqrt{a_2} P_2 + \cdots + \sqrt{a_k} P_k$.

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- This proves uniqueness. ■

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- For instance, for A positive, if $A = UDU^*$ is the diagonalization of A,

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  with any choice of sign on the diagonal is a square root of A.
- For any projection P, the unitary $2P - I = P - P^{\perp}$ is a square root of I. This shows that I has infinitely many square roots (in dimension bigger than 1) if we do not insist on positivity.

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- END OF LECTURE 27.