LINEAR ALGEBRA -II

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- Warning: A positive matrix need not have positive entries. It can have negative entries and also complex entries.
- Matrices whose entries are positive would be called as entrywise positive matrices. That is also an important class, but we will not be studying them now.

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- (v) $A = A^*$ and eigenvalues of A are non-negative.
- (vi) $A = S^2$ for some self-adjoint $n \times n$ matrix S.

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- (iii) $\Rightarrow$ (iv). We have $a_{ij} = \langle v_i, v_j \rangle, \quad \forall i, j$.
- Now for any $x \in \mathbb{C}^n$:

$$\begin{aligned} \langle x, Ax \rangle &= \sum_{i=1}^{n} \overline{x_i}(Ax)_i \\ &= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} a_{ij} x_j \\ &= \sum_{i=1}^{n} \overline{x_i}. \sum_{j=1}^{n} \langle v_i, v_j \rangle . x_j \end{aligned}$$

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- Therefore,

$$\begin{aligned}
\langle x, Ax\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x_i v_i, x_j v_j\rangle \\
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- First we want to show that $A = A^*$. Here we use the polarization identity and the fact that if $\langle v, w\rangle$ is real then $\langle v, w\rangle = \langle w, v\rangle$. For all x, y,

$$\begin{aligned}
\langle x, Ay\rangle &= \frac{1}{4}\sum_{j=0}^{3} i^{-j}\langle (x + i^j y), A(x + i^j y)\rangle \\
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- implies that $a \geq 0$ as $\langle x, x\rangle \neq 0$.

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- Take
$$S = U \begin{bmatrix} \sqrt{d_1} & 0 & \ldots & 0 \\ 0 & \sqrt{d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sqrt{d_n} \end{bmatrix} U^*.$$

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- Then clearly S is self-adjoint and $A = S^2$.

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- Clearly R is self-adjoint. We have the characteristic polynomial of R, as

$$p(x) = (x-2)^2 - 1 = x^2 - 4x + 3 = (x-1)(x-3).$$

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- Find all self-adjoint operators S such that $R = S^2$. (Exercise)

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- This proves uniqueness. ■
- This is known as Cartesian decomposition.

Constructing positive matrices out of positive matrices

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- Then by part (iv) of the characterization theorem, $aA + bB$ is positive. ∎

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- Note that this theorem does not follow directly from the definition of positivity or from the eigenvalue criterion.
- This theorem shows that the set of $n \times n$ positive matrices has 'cone' structure: It is closed under taking sums and it is closed under multiplication by positive scalar.

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- Proof: As A is positive, $A = D^*D$ for some matrix D.
- Now, $B^*AB = B^*D^*DB = (DB)^*(DB)$. Hence B^*AB is positive from the definition of positivity. We may also see this from looking at the quadratic form. $\blacksquare$

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- Proof: The first part is clear as the trace and determinant of a matrix are respectively the sum and the product of its eigenvalues and a positive matrix has non-negative eigenvalues. The second claim follows from $a_{ii} = \langle v_i, v_i \rangle$ in part (iv) of the characterization.

Gram matrices

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- We know that G is positive. Hence its determinant is positive. So we get $\langle x,x\rangle.\langle y,y\rangle - \langle x,y\rangle.\langle y,x\rangle \geq 0$.
- In other words, we have the Cauchy-Schwarz inequality:
$$|\langle x,y\rangle|^2 \leq \|x\|^2.\|y\|^2.$$

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- Let $b_1, b_2, \ldots, b_k$ the distinct eigenvalues of B and $B = b_1 Q_1 + b_2 Q_2 + \cdots + b_k Q_k$ be the spectral decomposition of B.

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- This proves uniqueness. $\blacksquare$

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  with any choice of sign on the diagonal is a square root of A.
- For any projection P, the unitary $2P - I = P - P^\perp$ is a square root of I. This shows that I has infinitely many square roots (in dimension bigger than 1) if we do not insist on positivity.

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- END OF REVIEW

Polar decomposition theorem

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- In particular, if $|B|x = |B|y$, then $|B|(x-y) = 0$, Hence,
$$\langle |B|(x-y), |B|(x-y)\rangle = 0.$$

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- Hence U_0 is isometric.
- From the definition of U_0 it is clear that U_0 maps $\mathcal{M}$ onto $\mathcal{N}$.

Continuation

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- Extend $\{v_1, v_2, \ldots, v_k\}$ to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$.

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- Now we extend U to a unitary of $\mathbb{C}^n$ by defining it on the orthonormal basis $\{v_1, \ldots, v_n\}$ (and extending linearly) by setting
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- Therefore, U is uniquely determined. ∎

Polar decomposition for normal matrices

- Example 28.2: Suppose A is a normal matrix and let

$$A = UDU^*$$

be the diagonalization of A with unitary U and diagonal matrix D. Let $d_1, d_2, \ldots, d_n$ be the diagonal entries of D.

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- We can write D as

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using the polar decompositions of $d_1, d_2, \dots, d_n$.

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- Note that if $d_j = 0$, then when we write $d_j = e^{i\theta_j}|d_j|$, $e^{i\theta_j}$ is not unique.

Continuation

- Now $A = V|A|$ where

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  is a unitary and

- $$|A| = U \begin{bmatrix} |d_1| & 0 & \dots & 0 \\ 0 & |d_2| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |d_n| \end{bmatrix} U^*$$

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Example

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- Recall that an $n \times n$ matrix with all entries equal to $\frac{1}{n}$ is a projection.

Example

- Example 28.3: Consider

$$B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Then

$$B^*B = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}.$$

- Recall that an $n \times n$ matrix with all entries equal to $\frac{1}{n}$ is a projection.
- Therefore

$$B^*B = 12 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} + 0. \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

is the spectral decomposition of B^*B.

Continuation

- Therefore,

$$
\begin{aligned}
|B| &= \sqrt{12}\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \\
&= \begin{bmatrix} 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \end{bmatrix}.
\end{aligned}
$$

Continuation

- Take
$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}.$$

Continuation

- Take
$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}.$$
- We may now verify that
$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} . \begin{bmatrix} 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \end{bmatrix}.$$
is a polar decomposition of B.

Continuation

- Take
$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}.$$
- We may now verify that
$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} . \begin{bmatrix} 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \end{bmatrix}.$$
is a polar decomposition of B.
- END OF LECTURE 28.