LINEAR ALGEBRA -II

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Lecture 29: Balanced incomplete block designs (BIBD)

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- Consider an agricultural experiment where we wish to determine the best combination of fertilizers and other inputs.
- Suppose out of 5 inputs at a time we can try out only 3 at a time.
- Two different inputs may interact with each other.
- In such situation it becomes helpful to use some combinatorial structures called balanced incomplete block designs.

BIBD

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- (iii) Each symbol appears in exactly r blocks.
- (iv) Each pair of symbols appear in exactly λ blocks.
- Here v, b, r, k, λ are natural numbers. It is also assumed $k < v$ (No block contains all the treatments). For this reason they are called incomplete designs.

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- Each pair of symbols appears in 2 (λ) blocks.

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- Blocks:

$$\{1,3,7,8\},\{1,2,4,8\},\{2,3,5,8\},\{3,4,6,8\},\{4,5,7,8\},$$

$$\{2,6,7,8\},\{1,2,3,6\},\{1,2,5,7\},\{1,3,4,5\},\{1,4,6,7\},$$

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Basic relations between parameters

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- (ii) Say the symbols are $\{1, 2, \ldots, v\}$. Consider how many times $(1, 2)$ appears in a block. It appears λ times. Similarly $(1, 3)$ appears λ times. So $(1, j)$ appears for some j, a total of $\lambda(v-1)$ times.

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- Since 1 has appeared in exactly r blocks, and each block has $(k-1)$ other elements, we have $(1, j)$ for some j appearing $r(k-1)$ times. This gives $r(k-1) = \lambda(v-1)$.

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- This is an important inequality and it is non-trivial!

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$$n_{ij} = \begin{cases} 1 & \text{if } i \in B_j \\ 0 & \text{otherwise} \end{cases}$$
- In other words, $n_{ij} = 1$ if i appears in the block B_j and it is zero otherwise.

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- The corresponding incidence matrix is given by:
- $$N = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$
- Note that

$$N^t = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad NN^t = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

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- Since every entry is 0 or 1, when $i = j$, $(NN^t)_{ii}$ is just the number of times the treatment i appears in the design. So

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- Proof of Fisher's inequality: Let P be the projection $\frac{1}{v}J$. Then

$$
\begin{aligned}
NN^t &= (r-\lambda)I + \lambda J \\
&= (r-\lambda)I + v\lambda P \\
&= (r-\lambda)(P + P^\perp) + v\lambda P \\
&= (r-\lambda)P^\perp + (r-\lambda+v\lambda)P.
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- Consequently $(r-\lambda) > 0$ and $(r-\lambda+v\lambda) > 0$.

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- END OF LECTURE 30.