# LINEAR ALGEBRA -II

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- Warning: A positive matrix need not have positive entries. It can have negative entries and also complex entries.
- Matrices whose entries are positive would be called as entrywise positive matrices. That is also an important class, but we will not be studying them now.

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- (vi) $A = S^2$ for some self-adjoint $n \times n$ matrix $S$.

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- Theorem 31.1: Suppose $A$ is a positive matrix and $a_{ii} = 0$ for some $i$. Then $a_{ij} = 0 = a_{ji}$ for all $j$, that is, if a diagonal entry is zero then all entries in corresponding row and column are zeros.

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- Proof: We write $A$ as a Gram matrix of some vectors $v_1, v_2, \ldots, v_n$.
- Now $a_{ii} = 0$ implies $\langle v_i, v_i \rangle = 0$ and hence $v_i = 0$.
- Consequently, $a_{ij} = \langle v_i, v_j \rangle = 0$ for any $j$. Similarly $a_{ji} = 0$ for any $j$.

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- Proof: Easy.

# Principal submatrices and principal minors

- Definition 31.3: Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be a complex matrix. Then for any non-empty subset $S$ of $\{1, 2, \dots, n\}$, the matrix $A_S := [a_{ij}]_{i,j \in S}$ is called the principal submatrix of $A$ corresponding to $S$. The determinant of $A_S$ is called the principal minor of $A$ corresponding to the subset $S$. The principal minors corresponding to sets of the form $\{1, 2, \dots, k\}$ for $1 \leq k \leq n$ are known as leading principal minors.

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- Theorem 31.4: Let $A$ be a positive matrix. Then all its principal minors are positive. If $A$ is strictly positive then all its principal minors are strictly positive.
- Proof: Exercise. (Hint: Use Gram matrices.)

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- Proof: We have already seen $(i) \Rightarrow (ii)$.
- To see the converse, we use induction on $n$.

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- Now for $n \geq 2$, assume the result for $(n-1)$ and we will prove it for $n$.
- Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be a matrix with all its leading principal minors strictly positive.
- In particular $a_{11} > 0$.

# Continuation

- We consider a block decomposition of $A$ as

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- Using the fact that $a_{11} > 0$, we have

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- where, $C = B - \frac{1}{a_{11}} yy^*$. (Recall ‘Schur-complement’.)
- Taking determinant, we get

$$\det(A) = a_{11}. \det(C).$$

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is also strictly positive.
- Further, as

$$A = \begin{bmatrix} 1 & 0 \\ \frac{y}{a_{11}} & I \end{bmatrix} \cdot \begin{bmatrix} a_{11} & 0 \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{y^*}{a_{11}} \\ 0 & I \end{bmatrix},$$

$A$ is positive. Since its determinant is non-zero it is strictly positive. ■

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- In other words, the previous theorem does not hold if strict positivity is replaced by positivity.
- To get the correct result for positivity we need to consider all principal minors instead of just leading principal minors.

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- There is nothing to show for $n = 1$.

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- If $a_{jj} = 0$ for every $j$, then by considering positivity of $2 \times 2$ principal minors we see that $a_{ij} = 0$ for all $i, j$. In other words, in such a case $A$ is the zero matrix and hence it is positive. ■

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- END OF LECTURE 31.