LINEAR ALGEBRA -II

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Lecture 35: Jordan canonical form

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- This is answered by Jordan canonical form theorem.

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- Notation: For $b \in \mathbb{C}$ and $n \in \mathbb{N}$, let $J_b(n)$ denote the $n \times n$ matrix whose diagonal entries are equal to b and the super diagonal entries are equal to 1 and all the other entries are equal to zero:

$$J_b(1) = [b], \quad J_b(n) = \begin{bmatrix} b & 1 & 0 & \dots & 0 \\ 0 & b & 1 & \dots & 0 \\ 0 & 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b \end{bmatrix}.$$

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- Alternatively,

$$(J_b(n))_{ij} = \begin{cases} b & \text{if} j = i; \\ 1 & \text{if } j = i+1; \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues and eigenvectors

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- Therefore the eigenvalue b has algebraic multiplicity n and geometric multiplicity 1 in $J_b(n)$.
- The characteristic polynomial of $J_b(n)$ is $(x - b)^n$.

Jordan block theorem

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- In other words, if A is an $n \times n$ matrix and $a_1, a_2, \ldots, a_k$ are the distinct eigenvalues of A, with geometric multiplicities $g_1, g_2, \ldots, g_k$, then there exist natural numbers $n_{ij}, \;\; 1 \leq i \leq k, 1 \leq j \leq g_i$ such that A is similar to

$$\oplus_{i=1}^{k} \oplus_{j=1}^{g_i} J_{a_i}(n_{ij}).$$

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- In other words, the sum of block sizes is equal to n and for each distinct eigenvalue there are as many Jordan blocks (possibly of different sizes) as the geometric multiplicity of the eigenvalue.

Examples

- Example 35.1: Suppose A is similar to

$$J_0(1) \oplus J_0(2) \oplus J_0(4) \oplus J_5(3).$$

Then the eigenvalues are 0 and 5. The order of A is $1 + 2 + 4 + 3 = 10$. The algebraic multiplicity of 0 is $(1 + 2 + 4) = 7$ and the geometric multiplicity of 0 is 3. The algebraic multiplicity of 5 is 3 and its geometric multiplicity is 1.

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- Note that if A is diagonalizable (in particular, if it is normal), all Jordan blocks would have size 1 and A is similar to

$$\oplus_{i=1}^{k} \oplus_{j=1}^{n_i} J_{a_i}(1).$$

  In other words geometric and algebraic multiplicities of a_i are equal to n_i and $n_1 + n_2 + \cdots + n_k = n$.

Minimal polynomial and characteristic polynomial

- Consider a Jordan block

$$J_b(n) = \begin{bmatrix} b & 1 & 0 & \dots & 0 \\ 0 & b & 1 & \dots & 0 \\ 0 & 0 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b \end{bmatrix}.$$

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- If we consider $B = J_b(n_1) \oplus J_b(n_2) \oplus \cdots J_b(n_r)$ then the characteristic polynomial of B is

$$p(x) = (x - b)^{n_1 + n_2 + \cdots + n_r}$$

and the minimal polynomial is $q(x) = (x - b)^m$ where $m = \max\{n_1, n_2, \dots, n_r\}$.

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- Thus we get the following result.

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- Theorem 35.1 Let A be a matrix similar to Jordan block

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as in Theorem 35.1. Then the characteristic polynomial of A is given by

$$p(x) = \Pi_{i=1}^{k}(x - a_i)^{\sum_{j=1}^{g_i} n_{ij}}$$

The minimal polynomial is given by

$$q(x) = \Pi_{i=1}^{k}(x - a_i)^{m_i},$$

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- and $N^n = 0$.

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- In other words,

$$(J_b(n)^m)_{ij} = \begin{cases} \left(\begin{array}{c} m \\ j-i \end{array} \right) b^{m-(j-i)} & i \leq j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

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- More explicitly: $(J_b(n))^m$ equals

$$\begin{bmatrix} b^m & mb^{m-1} & \binom{m}{2} b^{m-2} & \binom{m}{3} b^{m-3} & \cdots \\ 0 & b^m & mb^{m-1} & \binom{m}{2} b^{m-2} & \cdots \\ 0 & 0 & b^m & mb^{m-1} & \cdots \\ 0 & 0 & 0 & b^m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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- The proof of Jordan block theorem has been omitted.

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- END OF LECTURE 35.