# LINEAR ALGEBRA -II

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# Lecture 36: Simultaneous diagonalization

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- Theorem 36.1: Let $A_1, A_2$ be two commuting $n \times n$ complex matrices. Then there exists a unitary $U$ and two upper triangular matrices $T_1$, $T_2$ such that

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$$A_j = UT_jU^*, \quad j = 1, 2.$$

- Here we are considering the standard inner product on $\mathbb{C}^n$.
- The point is that when $A_1, A_2$ commute, we can find a single unitary $U$ such that both $U^*A_1U$ and $U^*A_2U$ are upper triangular.

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- Suppose $A_1 v = a_1 v$.
- By commutativity, $A_1 A_2 v = A_2 A_1 v = A_2(a_1 v) = a_1(A_2 v)$.
- In other words, the eigenspace $E_1 = \{v \in \mathbb{C}^n : Av = a_1 v\}$ is left invariant by $A_2$.

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- Let $\{v_1, v_2, \dots, v_k\}$ be an orthonormal basis for $E_1$.
- Extend it to an orthonormal basis $\mathcal{B} := \{v_1, \dots, v_n\}$ of $\mathbb{C}^n$.

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- Extend it to an orthonormal basis $\mathcal{B} := \{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$.
- Let $U_0$ be the unitary whose columns are $\{v_1, \ldots, v_n\}$.

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- for some $k \times (n-k)$ matrix $R_{12}$ and $(n-k) \times (n-k)$ matrix $R_{22}$ or equivalently,

$$A_1 U_0 = U_0 R \qquad (1)$$

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- From equations (1) and (2), we have

$$A_1 = U_0 R U_0^*, \quad A_2 = U_0 S U_0^*.$$

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- By block matrix computations,

$$\begin{bmatrix} a_1 S_{11} & a_1 S_{12} + R_{12} S_{22} \\ 0 & R_{22} S_{22} \end{bmatrix} = \begin{bmatrix} a_1 S_{11} & S_{11} R_{12} + S_{12} R_{22} \\ 0 & S_{22} R_{22} \end{bmatrix}.$$

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- In particular, $R_{22}$ and $S_{22}$ commute. Note that they have order $(n-k) \times (n-k)$ with $k \geq 1$. Hence the induction hypothesis is applicable.

# Continuation

- Therefore, there exist a unitary $W$, two upper triangular matrices $M_1, M_2$ (all of order $(n-k) \times (n-k)$), such that

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- We observe that,
$$\begin{aligned} A_1 &= U_0\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \cdot \begin{bmatrix} a_1 I_k & Z^*R_{12}W \\ 0 & M_1 \end{bmatrix}\begin{bmatrix} Z^* & 0 \\ 0 & W^* \end{bmatrix} U_0^* \\ &= U\begin{bmatrix} a_1 I_k & Z^*R_{12}W \\ 0 & M_1 \end{bmatrix} U^*, \end{aligned}$$

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- where
$$U = U_0 \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}$$
being a product of two unitaries is a unitary.

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- Similarly,

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\begin{aligned}
A_2 &= U_0 \begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix} \cdot \begin{bmatrix} X & Z^* S_{12} W \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} Z^* & 0 \\ 0 & W^* \end{bmatrix} U_0^* \\
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- $$T_1 := \begin{bmatrix} a_1 I_k & R_{12} \\ 0 & M_1 \end{bmatrix}$$

is upper triangular as $M_1$ is upper triangular.

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$$= U \begin{bmatrix} X & Z^* S_{12} W \\ 0 & M_2 \end{bmatrix} U^*,$$

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is upper triangular as $X$, $M_2$ are upper triangular. ■

# Simultaneous diagonalization

- Theorem 36.2: Suppose $A_1, A_2$ are commuting normal matrices. Then there exists a unitary $U$ with two diagonal matrices $D_1, D_2$ such that $A_1 = UD_1U^*, \quad A_2 = UD_2U^*$.

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- Corollary 36.3: Suppose $A_1, A_2$ are normal matrices. Then $A_1, A_2$ are commuting if and only if there exists a normal matrix $A$ with polynomials $p_1, p_2$ such that $A_1 = p_1(A), A_2 = p_2(A)$.

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- Proof: If $A_1, A_2$ are commuting, by the previous theorem, we may assume that both $A_1, A_2$ are diagonal. Now take $A$ as the diagonal matrix with $j$-th diagonal entry as $j$. It is easy to get polynomials $p_1, p_2$ so that $p_1(j) = (A_1)_{jj}, p_2(j) = (A_2)_{jj}$. Hence $p_1(A) = A_1, p_2(A) = A_2$.

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- The converse is to show that for any normal matrix $A$, $p_1(A), p_2(A)$ commute for any two polynomials and is easy.

# Families of commuting matrices

- Theorem 36.4: Fix $k \geq 1$. Suppose $A_1, A_2, \ldots, A_k$ are commuting matrices. Then there exists a unitary $U$ with upper triangular matrices $T_1, \ldots, T_k$ such that

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- Theorem 36.5: Fix $k \geq 1$. Suppose $A_1, A_2, \ldots, A_k$ are commuting normal matrices. Then there exists a unitary $U$ with diagonal matrices $D_1, \ldots, D_k$ such that

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- Proof: Clear from the previous theorem.

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- This implies $A_1^2S = SA_1^2$. Hence $S^{-1}A_2S = A_2$. But $A_2$ is not in Jordan form.

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- END OF LECTURE 36.