LINEAR ALGEBRA -II

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Lecture 37: Real symmetric matrices and quadratic forms

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- We can't diagonalize A or make it upper triangular in real field.

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- This is answered by the following theorems.

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- Theorem 37.1: Let A be a real matrix with only real eigenvalues. Then there exists an orthogonal matrix M and an upper triangular matrix T such that

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- Proof: Suppose $v \in \mathbb{C}^n$ is non-zero and $Av = dv$.
- Let $v = x + iy$, where x, y are real vectors.
- From $Av = dv$, we get $A(x + iy) = dx + idy$. Since A has real entries and d is real, by comparing the real and imaginary parts we get $Ax = dx$ and $Ay = idy$. As $v \neq 0$, at least one of x or y is non-zero and take that as w. ■.

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- Just observe that the matrices of lower order appearing in the induction hypothesis also have real entries and real eigenvalues.

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- So the previous theorem is applicable.
- Now the result is immediate as symmetric upper triangular matrices are diagonal. ■

Jordan Canonical form

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- The proof of this is omitted.

Real quadratic forms

- Definition 37.5 Fix $n \in \mathbb{N}$. An n-variable quadratic form Q is a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^{n} b_i x_i^2 + \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j .$$

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- Note that here we are considering standard inner product on $\mathbb{R}^n$.

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- $$\begin{aligned} \langle x, Ax \rangle &= \sum_{i,j=1}^{n} x_i a_{ij} x_j \\ &= \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j \\ &= \sum_{i=1}^{n} b_i x_i^2 + \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j \\ &= Q(x). \end{aligned}$$

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- The uniqueness is clear from comparison of coefficients. ■.

Diagonalization

- Consider a real quadratic form Q. With out loss of generality, we may take

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for some real symmetric matrix A.

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- Then,

$$Q(x) = \langle x, MDM^{-1}x \rangle = \langle y, Dy \rangle.$$

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- We get,

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