Indian Statistical Institute

HW - 1: Analysis of Several Variables. Due date: 16/08/2022 Instructor : Jaydeb Sarkar

---

NOTE: (i) $B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$. (ii) $D_r(a) = B_r(a) \setminus \{a\}$. (iii) $S \subseteq \mathbb{R}^n$.

(1) Prove that $|\|x\| - \|y\|| \leq \|x - y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.

(2) Compute the limit points of

$$(i)\ B_r(a), \qquad (ii)\ \{x \in \mathbb{R}^n : \|x\| = 1\}, \qquad (iii)\ \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x, y \in (0, 1)\}.$$

(3) Discuss the following limits:

$$(i) \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}.\ (ii) \lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y)}{x^2 + y}.\ (iii) \lim_{(x,y)\to(0,0)} \frac{x^2 y^3}{x^4 + y^6}.$$

(4) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $f : \mathbb{R}^2 \to R$ by $f(x, y) = g(xy)$. Is f continuous?

(5) Which of the following functions on $\mathbb{R}^2$ can be defined continuously at $(0, 0)$?

$$(i) f(x, y) = \begin{cases} \frac{x^2 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \qquad (ii) f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^6} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(6) Can the function $f(x, y) = \frac{xy}{|x| + |y|}$ be extended to a continuous function on $\mathbb{R}^2$?

(7) Prove that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, where

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

(8) Prove that $f : S \to \mathbb{R}^m$ is uniformly continuous if and only if $\Pi_i f$ (the i-th projection) is uniformly continuous for all $i = 1, \ldots, m$.