Indian Statistical Institute

HW - 2: Analysis of Several Variables. Due date: 29/08/2022 Instructor : Jaydeb Sarkar

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NOTE: $\mathcal{O}_n \subseteq \mathbb{R}^n$ open subset.

(1) Discuss the differentiability of $f : \mathbb{R}^n \to \mathbb{R}$, where $f(x) = \sum_{i=1}^n |x_i|$.

(2) Prove that if f is a function that is differentiable on a convex set $\mathcal{O}_2$, and has bounded partial derivatives on $\mathcal{O}_2$, then f is uniformly continuous on $\mathcal{O}_2$.

(3) Compute the derivative of f at $(0, -\ln 6, \ln 2, 1)$ (if exists), where

$$f(x, y, z, w) = (xe^y, ze^{-w}, e^x).$$

(4) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^2y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f has first order partial derivatives on $\mathbb{R}^2$. Is f continuously differentiable on $\mathbb{R}^2$ (that is, $f \in C^1(\mathbb{R}^2)$)?

(5) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x^3 \sin \frac{1}{x} + y^2 & \text{if } x \neq 0 \\ y^2 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable at $(0,0)$ and f_x is not continuous at $(0,0)$.

(6) Let $g : \mathcal{O}_n \to \mathbb{R}$ has first-order partial derivatives, and suppose

$$\frac{\partial f}{\partial x_i} \equiv 0 \qquad (i = 1\ldots, n),$$

on $\mathcal{O}_n$. Prove that there exists a constant c such that $f \equiv c$.

(7) For what differentiable mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ is $[(Df)(a)]$ a diagonal matrix (w.r.t. standard bases) for all $a \in \mathbb{R}^n$?

(8) Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(x, y, z) = (x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3).$$

Compute $J_f(x, y, z)$ (the Jacobian of f at (x, y, z)). Prove that the Jacobian is nonsingular unless two of the three variables are equal. Describe the locus of the singularities.

(9) True of False (with justification): if $f : \mathcal{O}_n \to \mathbb{R}^m$ is a differentiable function and $(Df)(x) = 0$ for all $x \in \mathcal{O}_n$, then f is a constant function.

(10) Let $f : (a, b) \to \mathbb{R}^n$ be a differentiable function. Prove that

$$\frac{d}{dt}(\|f(t)\|^2) = 2\langle f(t), Df(t)\rangle \qquad (t \in (a, b)).$$