Indian Statistical Institute

HW - 4: Analysis of Several Variables. Due date: 24/10/2022   Instructor : Jaydeb Sarkar

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(1) Fix $n \in \mathbb{N}$. Consider the partition $P = P^1 \times P^2 \in \mathcal{P}([0,1] \times [0,2])$, where

$$P^1 : \frac{j}{n},\ 0 \le j \le n, \text{ and } P^2 : \frac{2j}{n},\ 0 \le j \le n.$$

Compute $L(f, P)$, where $f(x,y) = x + y$.

(2) Compute $\int_{B^2} x^2 y^3 dx\, dy$, where $B^2 = [a_1, b_1] \times [a_2, b_2]$.

(3) Compute $\int_0^1 \int_{-1}^1 x e^{xy} dx dy$.

(4) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{q} & \text{if } (x,y) \in \mathbb{Q} \times \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise.} \end{cases}$$

(i) Prove that $f \in R([0,1] \times [0,1])$. (ii) Compute $\overline{\int_0^1} f(x,y) dx$ and $\underline{\int_0^1} f(x,y) dx$ for all $y \in [0,1]$. Prove that they are unequal for all $y \in \mathbb{Q}$. (iii) Prove that $\int_0^1 \int_0^1 f(x,y) dy dx$ exists, but $\int_0^1 \int_0^1 f(x,y) dx dy$ does not.

(5) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } \frac{1}{2^n} \le x, y \le \frac{1}{2^{n-1}} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $f \in R([0,1] \times [0,1])$. Also compute $\int f$.