Indian Statistical Institute

HW - 5: Analysis of Several Variables. Due date: 31/10/2022 Instructor : Jaydeb Sarkar

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(1) Show that the set $\{(x, y) : x^2 + 4y^2 = 16\}$ has content zero.

(2) Suppose $f : [0,1] \times [0,1] \to \mathbb{R}$ has continuous second partial derivatives. Suppose $f(0,0) = 1, f(0,1) = 2, f(1,0) = 3$ and $f(1,1) = 5$. Find

$$\iint_{[0,1]\times[0,1]} \frac{\partial^2 f}{\partial x \partial y}.$$

(3) Evaluate

$$(i) \iint_{[0,1]\times[1,2]} \frac{1}{2x+y}, \ (ii) \iint_{[1,2]\times[1,2]} \ln(x+y), \ (iii) \iint_{[0,1]\times[0,1]} x \exp(yx).$$

(4) Evaluate

$$(i) \int_0^{\frac{\pi}{2}} \int_0^{\cos y} x \sin y \, dx \, dy, \ (ii) \int_{-1}^{1} \int_0^{|x|} dy \, dx, \ (iii) \int_0^2 \int_1^3 |x-2| \sin y \, dx \, dy.$$

(5) Let $f \in C([0,1])$. Prove that

$$\Big(\int_0^1 f(x)\, dx\Big)^2 \leq \int_0^1 \Big(f(x)\Big)^2 dx.$$

[Hint: Consider $\int_0^1 \int_0^1 \Big(f(x) - f(y)\Big)^2 dy \, dx$.]

(6) Evaluate $\iint_\Omega \sin(y^2)$, where $\Omega$ is the triangle bounded by the lines $x = 0$, $y = x$, and $y = \sqrt{\pi}$.

(7) Reverse the order of integration $I = \int_1^2 \int_{2-x}^{\sqrt{2x-x^2}} f(x,y) dy \, dx$, that is, express $I$ as $\int_?^? \int_?^? f(x,y) dx \, dy$.

(8) Evaluate $\iint_\Omega |xy|$, where $\Omega$ is the disk of radius 1 centered at the origin.

(9) Evaluate $\iint_\Omega x^3 \exp(y^3)$, where $\Omega = \{(x,y) : 0 \leq x \leq 3, x^2 \leq y \leq 9\}$.

(10) Evaluate $\int_\Omega 3x^2 + 2y + z$, where

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : |x-y|, |y-z|, |x-z| \leq 1\}.$$

(11) Evaluate the volume of the solid bounded by the planes $x = 0$, $y = 0$, and $z = 0$, and $x + y + z = 1$.

(12) Find the area of the region bounded by $y = x$ and $y = x^2$.

(13) Find the volume of the region in $\mathbb{R}^3$ lying above the triangle with vertices $(-1,0), (0,1)$, and $(1,0)$ and under the graph of the function $f(x,y) = x^2 y$.

(14) If $\Omega = \overline{B_1((0,0))}$, then prove that

$$\frac{\pi}{3} \leq \int_\Omega \frac{1}{\sqrt{x^2 + (y-2)^2}} \leq \pi.$$

(15) Prove that the volume of the solid ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi abc}{3}$.

(16) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the $xy$-plane, and inside the cylinder $x^2 + y^2 = 2x$.

(17) Compute $\int_\Omega \frac{y}{x} dA$, where $\Omega$ is the region bounded by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = \frac{1}{2}x$.

(18) Compute $\int_\Omega \exp(\frac{x-y}{x+y})dA$, where $\Omega = \{(x,y) : x, y \geq 0, x+y \leq 1\}$.
[Hint: Use the substitution: $u = x + y$ and $v = x - y$.]

(19) Find the volume generated by the cone $z = \sqrt{x^2+y^2}$ and $0 \leq z \leq 3$.

(20) Compute $\int_0^1 \int_0^{\sqrt{x}} y \exp(\sqrt{x}) dy\, dx$. [Hint: Use $x \mapsto x^2$ and $y \mapsto y$.]

(21) Evaluate $\int_0^1 \int_0^z \int_0^y \exp\left((1-x)^3\right) dx\, dy\, dz$.

(22) Evaluate

$$\int\limits_{x^2+y^2+z^2\leq 1} \exp\left((x^2+y^2+z^2)^{\frac{3}{2}}\right).$$