Physics I

Lecture 1

Text Book :

Thornton and Marion : Classical Dynamics of Particles and Fields

- John Taylor : Classical Mechanics
- Gregory : Classical Mechanics
- Morin : Classical Mechanics (Lots of Problems)

Marks Distribution:

Homework : 30

Quizzes :30

Final : 40

Classical Mechanics

→ Newtonian Mechanics and a bit of foray into reformulation by Lagrange and Hamilton

larger picture : How Newtonian Dynamics fits into rest of Physics

Basic Question : studying particles/bodies in motion → planets, balls, ... atoms

Greeks : Aristotle

2000 yrs ago

Galileo (1564 - 1642) experiment ⟶ laws.

Newton (1642 - 1727) Expt + mathematical formulation

↓ starting point.

Classical Mechanics

Newton ($\vec{F} = m\vec{a}$).

explained planets, apples, tides! 200+ years

No experimental contradictions!!

20th Century

breaks down for $v/c \simeq 1$

Einstein → Special Relativity
(1905).

breaks down for very small objects → atoms, subatomic particles

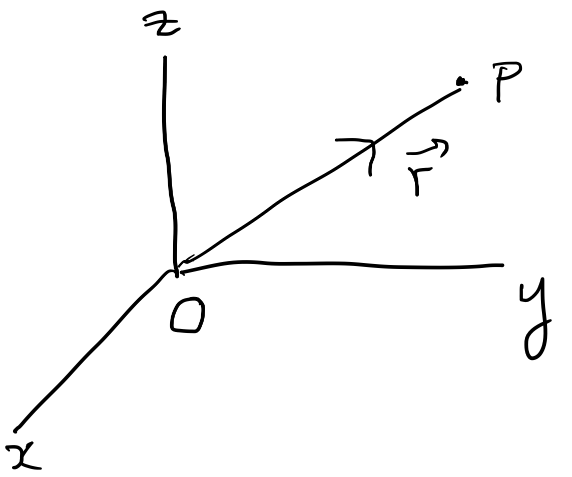
Quantum Mechanics

↳ Reduce to Newton in appropriate limits.

Newtonian dynamics has a wide range of applicability

Reference frames / coordinate systems

- To describe dynamics need to specify location of a particle
- Must specify coordinate system



$\vec{r}$ (depends on choice of origin)

$\vec{r} = (x, y, z)$ Cartesian components

vector $\vec{r}$: geometrical object

components are not.

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\equiv x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \sum_i r_i \hat{e}_i \quad \hat{e}_1 = \hat{x},\ \hat{e}_2 = \hat{y},\ \hat{e}_3 = \hat{z}$$

<u>Time</u> : In Newtonian dynamics time is absolute quantity. Does not depend on reference frame. All observers agree on time measured. Only freedom is in choice of origin of time.

Reference frame :

Every problem in classical mech is formulated w.r.t a specific ref. frame ; $\longrightarrow$ choice of spatial axes and origin.

- An important diff arises when two ref frames are in relative motion.

Newton's Laws

1st Law : In absence of external forces, a particle moves with constant vel $\vec{v}$.

Second Law : $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$ $\vec{p}$: momentum

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$$

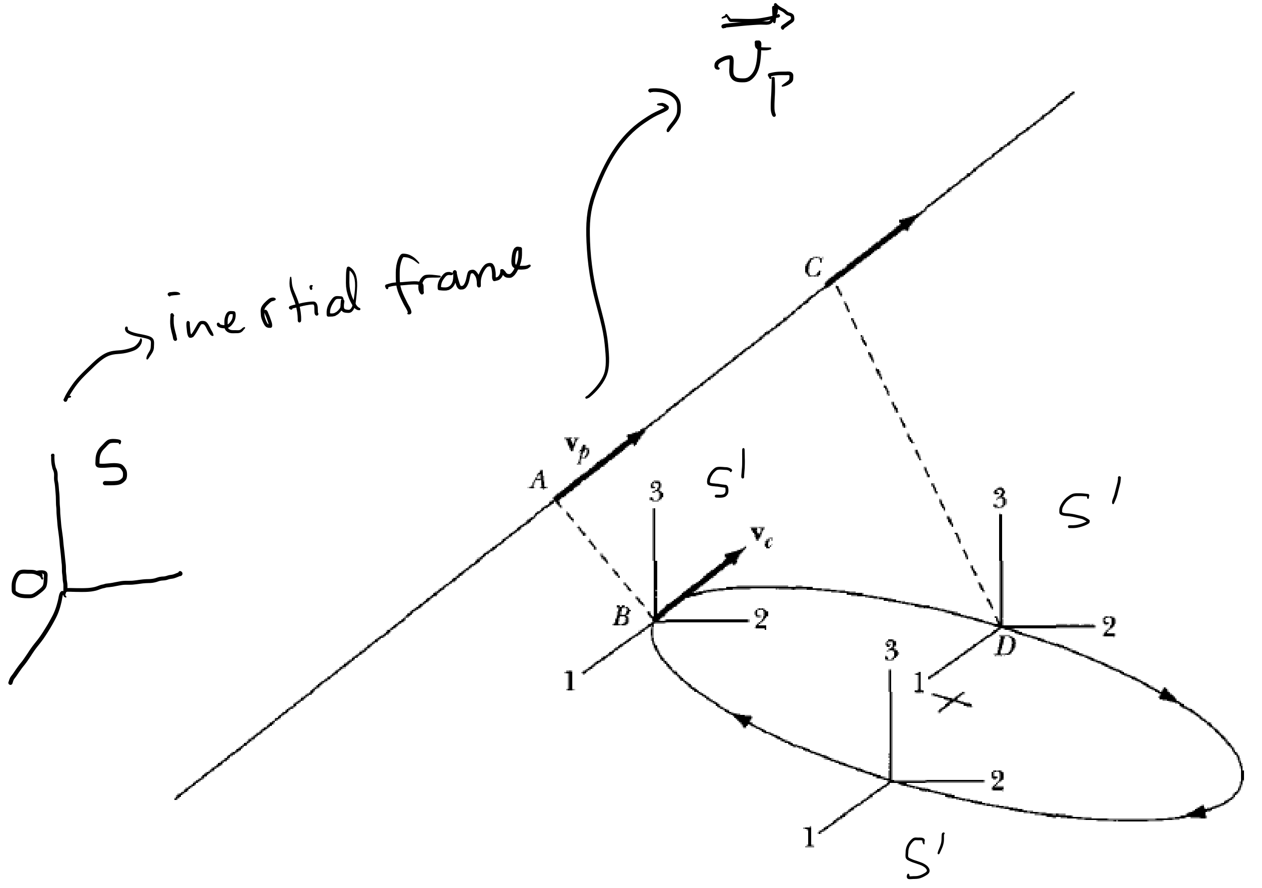
$$\boxed{\vec{F} = m\ddot{\vec{r}}} \longrightarrow \text{given initial conditions solve for } \vec{r}(t).$$

- Is the 1st Law merely a special case of second law?

  ↓ related to issue of ref frames. Newton's Law does not hold for all ref frames!

The special class of ref. frames in which the first law holds ~~are~~ is called <u>inertial reference frames</u>.

↳ 1st Law ⟹ inertial frames exist in nature



What does S' observe?

when particle is at A in frame S moving with $\vec{v}_p$

S' measures $\vec{v}_c = \vec{v}_p$

S' will observe particle particle at rest at A when S' is at B.

When particle is at C

S measures vel $\vec{v}_p$

At C, $\vec{S'}$ at D will observe particle moving with non-zero velocity!

So S observes particle with uniform vel.

S' observes particle accelerating.

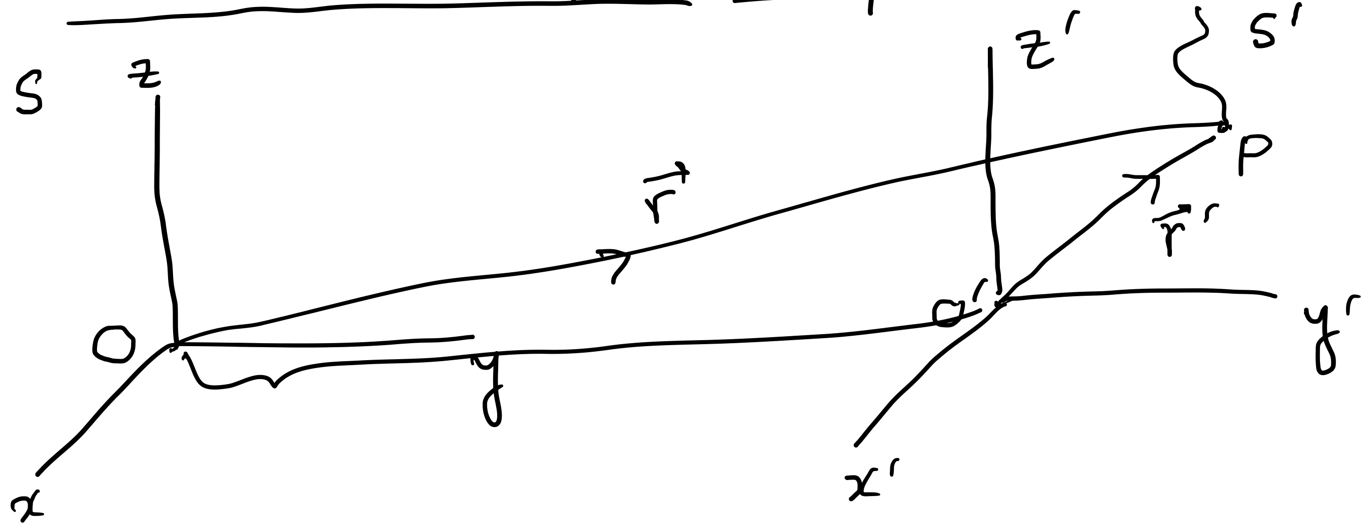
when there is no force!

Physics I

Lecture 2

Newton's Laws hold in inertial frames.

Galilean transformations / Galilean invariance



S' moves with uniform vel u along y

at $t=0$, O & O' coincided

$$\vec{r} = \vec{r}\,' + \vec{u}t \qquad \{ t' = t \} \rightarrow \text{implicit assumption}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}\,'}{dt} + \vec{u} \longrightarrow \boxed{\vec{v} = \vec{v}\,' + \vec{u}}$$

$$\left.\begin{aligned}\vec{r} &= \vec{r}\,' + \vec{u}t \\ \vec{v} &= \vec{v}\,' + \vec{u}\end{aligned}\right\} \longrightarrow \vec{r}, \vec{v} \text{ are not absolute but relative.}$$

$$\frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt}$$

$$\boxed{\vec{a} = \vec{a}\,'} \longrightarrow \text{acceleration is absolute}$$

$$\underbrace{\vec{F} = m\vec{a}}$$

m is scalar and is frame independent.

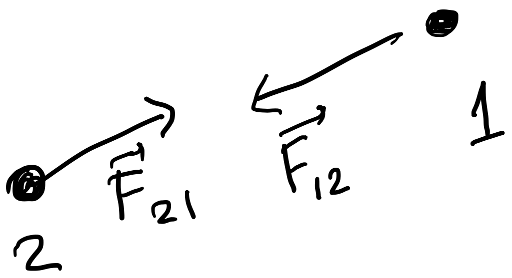
$\downarrow$ holds in all frames

$\longrightarrow$ Invariant under Galilean transformation

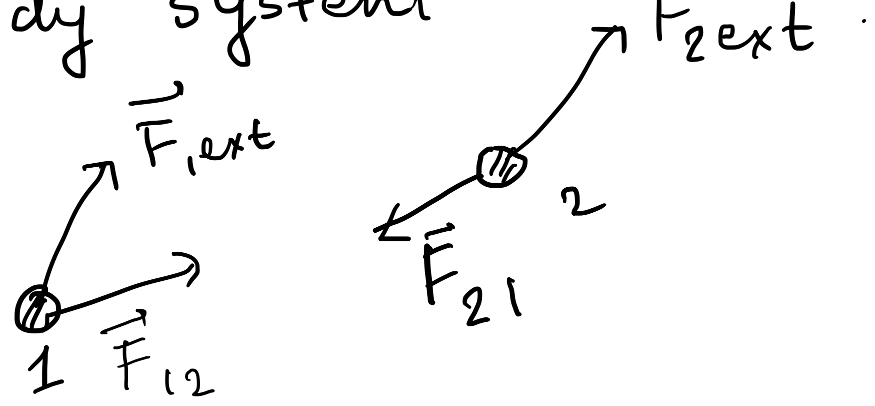
Newton's 3rd Law

If object 1 exerts force $\vec{F}_{21}$ on object 2 then object 2 always exerts a reaction force $\vec{F}_{12}$ on 1 such that

$$\boxed{\vec{F}_{21} = -\vec{F}_{21}}$$



2 body system



$$\dot{\vec{p}}_1 = \vec{F}_1 = \vec{F}_1^{ext} + \vec{F}_{12}$$

$$\dot{\vec{p}}_2 = \vec{F}_2 = \vec{F}_2^{ext} + \vec{F}_{21}$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \left(\vec{F}_1^{ext} + \vec{F}_2^{ext} + \underbrace{\vec{F}_{12} + \vec{F}_{21}}_{=0}\right)$$

$$\boxed{\dot{\vec{p}} = \vec{F}^{ext}}$$

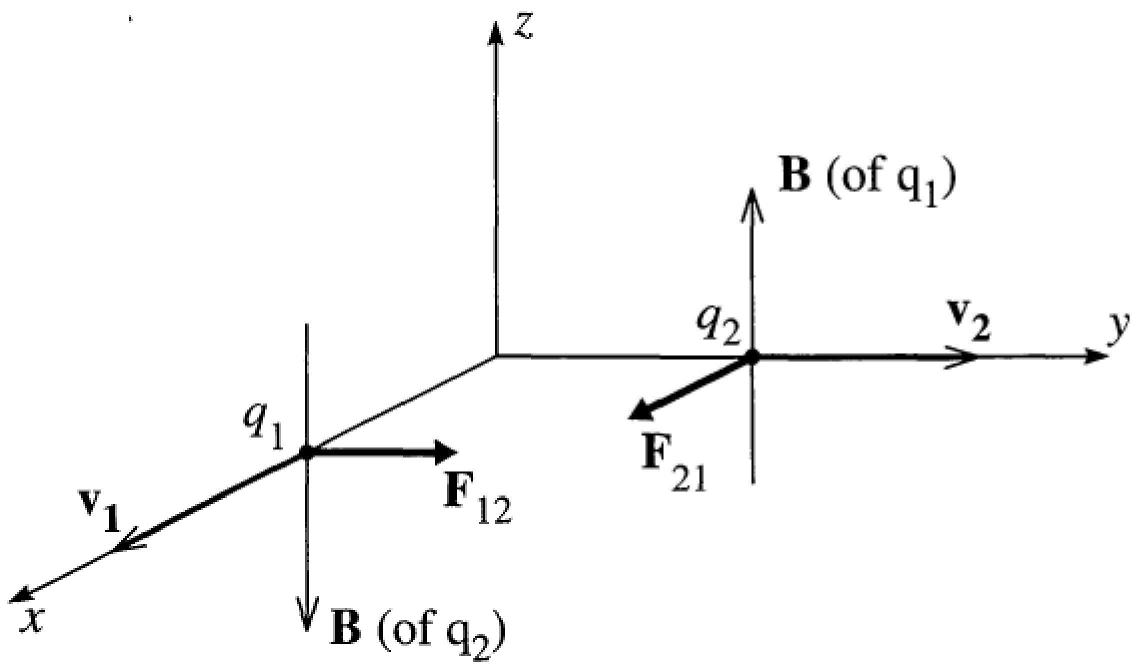
If $\vec{F}^{ext} = 0$

$$\dot{\vec{p}} = 0$$

$$\Rightarrow \boxed{\vec{p} = \text{const}} \longrightarrow \text{conservation of momentum}$$

$\vec{F}_{12} \neq \vec{F}_{21}$!

third law violated.



Is 3rd Law always true?

Relativity → time is not universal / observer dependent

$$\vec{F}_{12}(t) = -\vec{F}_{21}(t)$$ → cannot hold true for all observers

measured at same time

simultaneity is NOT ABSOLUTE

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Lecture 3

Third Law

$$\vec{F}_{12} = -\vec{F}_{21}$$

strong form

Force acts along line joining two particles.

central forces.

$$\boxed{\vec{p}_1 + \vec{p}_2 = \vec{P}}$$

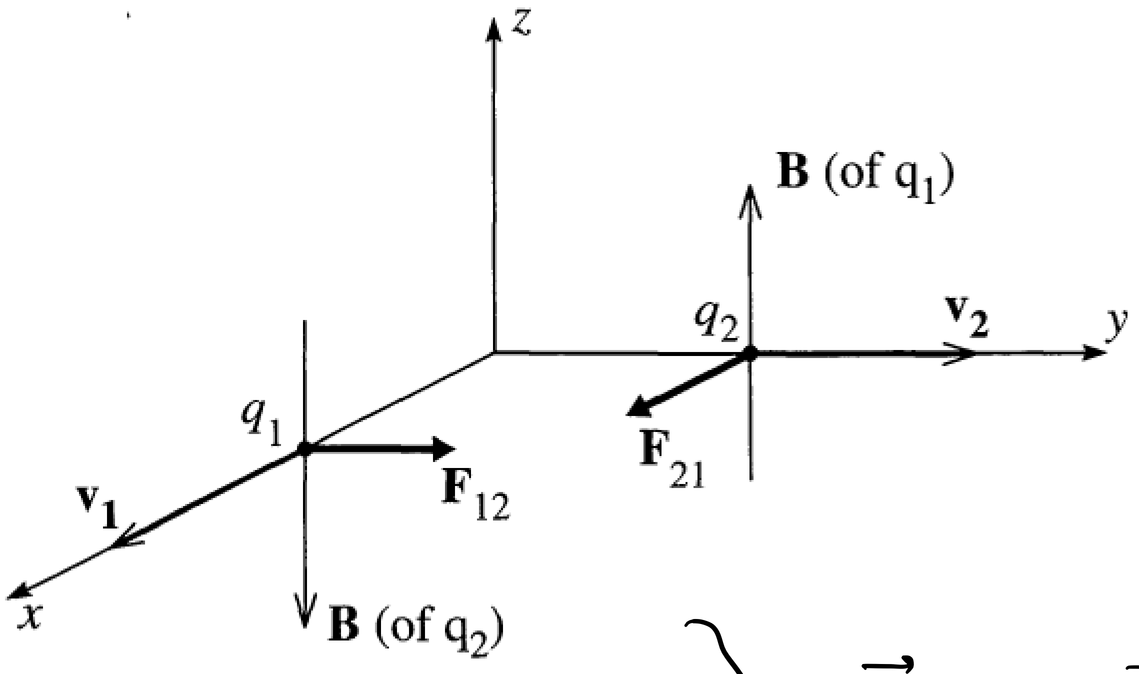
$$\boxed{\vec{P} = \text{const} \quad \text{when } \vec{F}_{ext} = 0}$$

Conservation of linear momentum

$\vec{F}_{12}(t) = -\vec{F}_{21}(t)$

Third Law is not always valid

↓ moving charges mag fields



$\vec{F}_{mag} = q(\vec{v} \times \vec{B})$

$\vec{F}_{12} \neq -\vec{F}_{21}$

Coulomb force $q\vec{E}$ is central
acts along line joining charges
obeys third law.

↳ Mom conservation is violated!!

Electromag fields carry momentum

Turns ~~or~~ out that

$\boxed{\text{Particle + field momentum is indeed conserved.}}$

↓ full theory of electromagnetism

$v \ll c$ violation is negligible.

mag field contribution is $\frac{v^2}{c^2}$ (Coulomb force)

Basic Problem

$$\vec{F} = m\ddot{\vec{r}}$$

3 2nd order diff eqn.s, in principle coupled.

integrate to find

$\vec{r} = \vec{r}(t)$, when $\vec{F}(\vec{r}, \vec{v}, t)$ is known

given initial conditions

$\vec{r}(0) = \vec{r}_0$

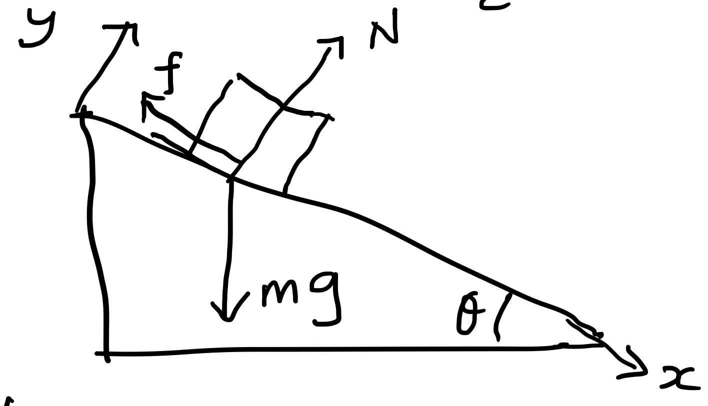
$\vec{v}(0) = \vec{v}_0$

In cartesian coordinates

$$m\ddot{x} = F_x$$

$$m\ddot{y} = F_y$$

$$m\ddot{z} = F_z$$

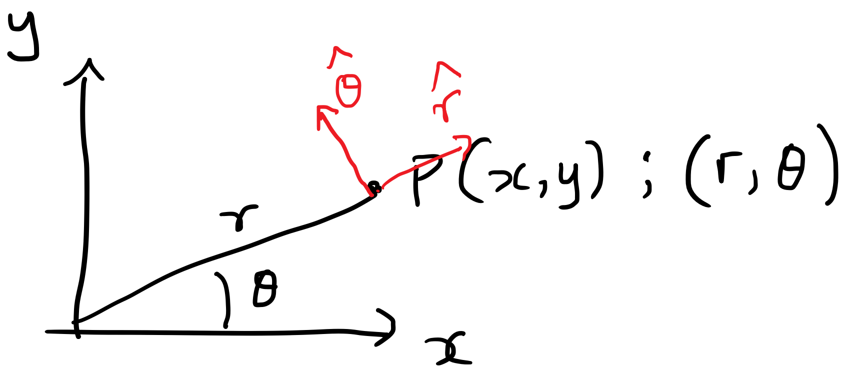


$x(t) = ?$

$y(t) = ?$

↳ sample problem

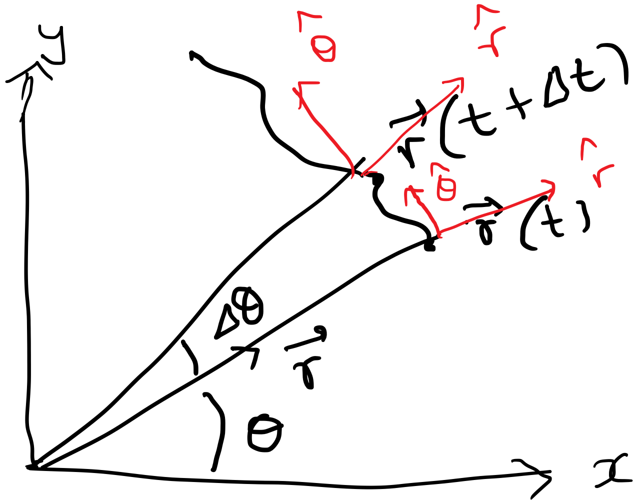
2D polar coordinates

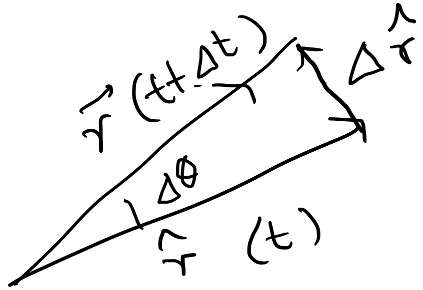


$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$r = \sqrt{x^2+y^2} \;.$$

$$\vec{r} = x\,\hat{x} + y\,\hat{y}$$
$$\vec{r} = r\,\hat{r}$$

$$\vec{F} = F_r\,\hat{r} + F_\theta\,\hat{\theta} \;.$$





$$\Delta\hat{r} \simeq \Delta\theta\,\hat{\theta}$$

$$\lim_{\Delta t\to 0}\frac{\Delta\hat{r}}{\Delta t} \simeq \frac{\Delta\theta}{\Delta\theta}\,\hat{\theta}$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}}$$

$$\dot{\vec{r}} = \frac{d}{dt}(r\,\hat{r})$$

$$\vec{V} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \underbrace{\dot{r}}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\boxed{\begin{aligned}\vec{v} &= v_r\,\hat{r} + v_\theta\,\hat{\theta}\\ &= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}\end{aligned}}$$

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

Alternatively,

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y}$$

$$= \dot{\theta}\left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}} \qquad \boxed{\dot{\hat{\theta}} = -\dot{\theta}\,\hat{r}}$$

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}\right).$$

$$= \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}^2\,\hat{r}$$

$$\boxed{\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$$

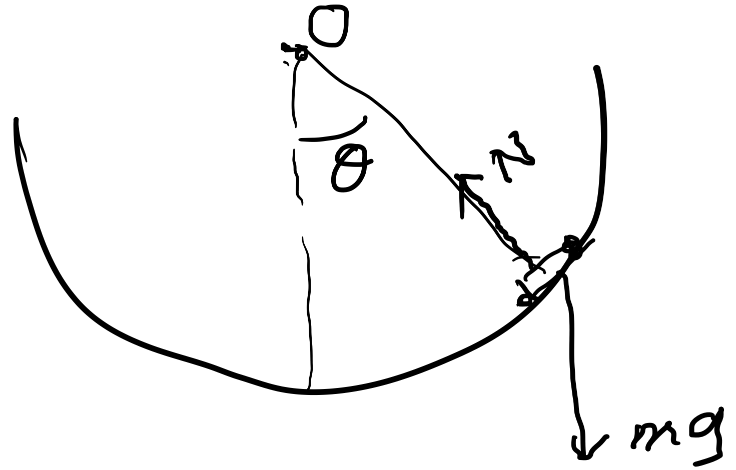
$$\boxed{\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{aligned}}$$

$$F_x = m\ddot{x}$$

$$F_y = m\ddot{y}$$

Oscillating skateboard

skateboard released short way from bottom
how long will it take to come back to same position



$r = R$

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = -mR\dot{\theta}^2$$

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mR\ddot{\theta}$$

$$F_r = mg\cos\theta - N.$$

$$F_\theta = -mg\sin\theta.$$

$$\boxed{mR\ddot{\theta} = -mg\sin\theta}$$

$$-mg\sin\theta = mR\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

small angle

$$\boxed{\ddot{\theta} + \frac{g}{R}\theta = 0}$$

$\theta = 0$, $\dot{\theta} = 0$, $\ddot{\theta} = 0$

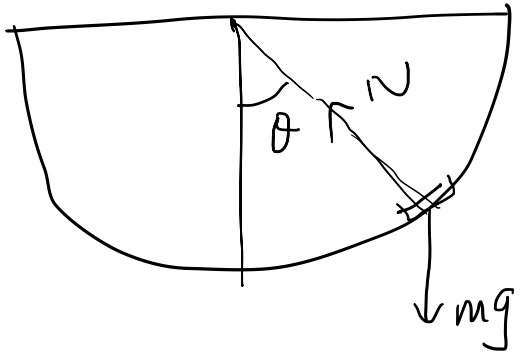
→ equilibrium position -

Physics I

Lecture 4

Recap

$$\vec{F} = m\vec{a}$$



$$\ddot{\theta} = -\frac{g}{R}\sin\theta.$$

$$\ddot{\theta} \simeq -\frac{g}{R}\theta.$$

equilibrium position

$\ddot{\theta} = 0, \theta = 0$

$\theta > 0, \ddot{\theta} < 0$

$\theta < 0 \quad \ddot{\theta} > 0$

$$\ddot{\theta} + \frac{g}{R}\theta = 0$$

$$\frac{g}{R} = \omega^2$$

$$\theta(t) = A\sin\omega t + B\cos\omega t$$

$t=0, \theta = \theta_0, \dot{\theta} = 0.$

$$\boxed{\theta(t) = \theta_0\cos\omega t}$$

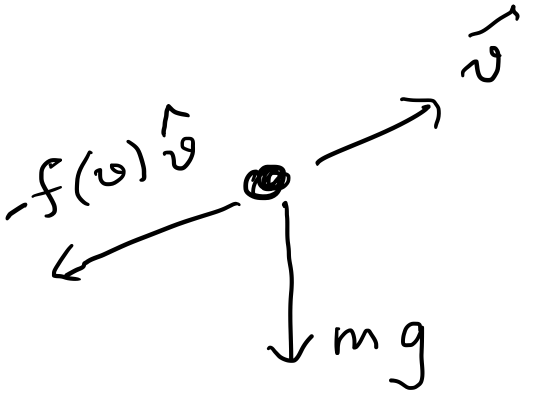
$$\vec{F} = m\vec{a}$$

$\hookrightarrow \vec{F}(v, r, t)$

Projectile motion with air resistance

Retarding forces

$$\vec{F} = \vec{F}(\vec{v}) = -f(v)\hat{v}$$



At low speeds

$$f(v) = bv + cv^2 = f_{lin} + f_{quad}$$

$f_{lin} \rightarrow$ viscous drag of medium $\propto$ viscosity of medium
depends on the size of the particle.

Stokes Law $f = 6\pi r \eta v$

$f_{quad} \rightarrow$ projectiles need to accelerate mass of air which they are in contact with; continuously colliding $\propto$ density of medium and cross sectional area.

for a spherical projectile

$$b = \beta \underbrace{D}_{\text{diameter}} \quad , \quad = \gamma D^2$$

for air $\quad \beta = 1.6 \times 10^{-4} \frac{N.s}{m^2}$

$$\gamma = 0.25 \frac{N.s}{m^2}$$

$$\frac{f_{quad}}{f_{lin}} = \frac{c v^2}{b v} = \frac{\gamma D}{\beta} v = \left(1.6 \times 10^3 \frac{s}{m^2}\right) D v$$

cricket ball vs raindrop

$D = 7\,cm$ $\quad v = 5\,m/s$ $\qquad D = 1\,mm$ $\quad v = 0.6\,m/s$.

Milikan oil drop

$D = 1.5\,\mu m$

$v = \quad \times 10^{-5}\,m/s$

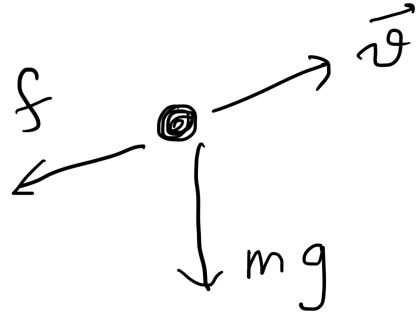
$$\frac{f_{quad}}{f_{lin}} \approx 600 \quad \text{cricket ball}$$

$$\frac{f_q}{f_l} \sim 1 \quad \text{raindrop}.$$

$$\frac{f_q}{f_l} \approx 10^{-7}$$

$$\vec{f} = -b\vec{v}$$

Linear air resistance



$$\vec{F} = m\vec{g} - b\vec{v} \quad (1)$$

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v} \quad (2)$$

$$m\dot{v}_x = -bv_x \quad (3)$$

$$m\dot{v}_y = mg - bv_y \quad (4)$$

for quadratic drag. $\vec{f} = -cv^2\hat{v} = -cv\vec{v}$

$$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x$$

$$m\dot{v}_y = mg - c\sqrt{v_x^2 + v_y^2}\, v_y$$

Horizontal motion with linear drag

at $t = 0$, $x = 0$, $v_x = v_{x0}$.

$$m \dot{v}_x = -b v_x$$

$$\dot{v}_x = -k v_x \qquad \boxed{k = \frac{b}{m}}$$

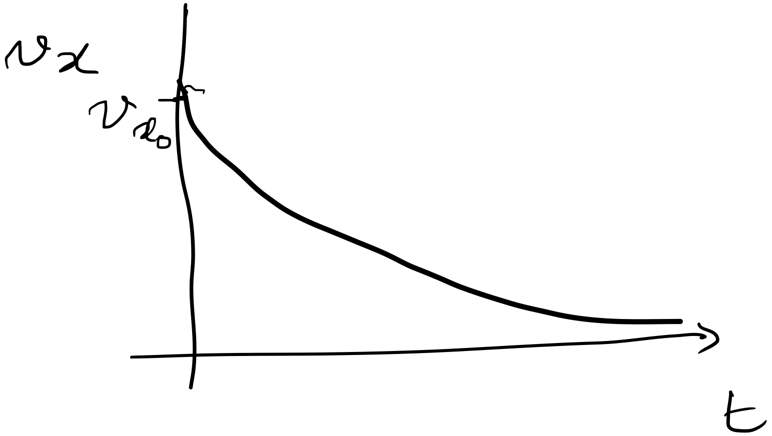
$$\frac{\dot{v}_x}{v} = -k$$

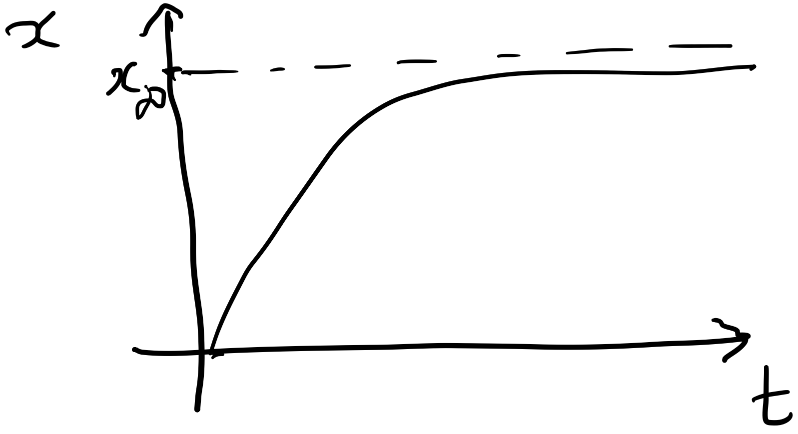
$$\frac{dv}{v} = -k\,dt$$

$$\boxed{v_x = A e^{-kt}} + \text{initial conditions.}$$

$$\boxed{v_x = v_{x0} e^{-kt} = v_{x0} e^{-t/\tau}} \qquad \tau = \frac{1}{k} = \frac{m}{b}.$$

$$v_x(t \to \infty) = 0$$





$$\frac{dx}{dt} = v_{x_0} e^{-kt}$$

$$x = -\frac{v_{x_0}}{k} e^{-kt} + C \;, \; t = 0 \;, \; x = 0 \Longrightarrow C = \frac{v_{x_0}}{k}$$

$$\boxed{x = \frac{v_{x_0}}{k}\left(1 - e^{-t/\tau}\right)}$$

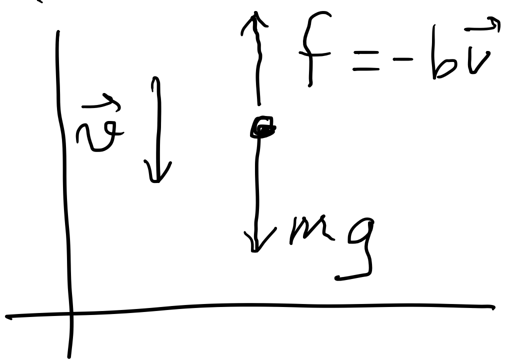
$$x_\infty = \frac{v_{x_0}}{k}$$

short time limit

$$x \simeq \frac{v_{x_0}}{k}\left(1 - 1 + \frac{t}{\tau}\right)$$

$$\simeq v_{x_0} t \, .$$

Vertical motion with linear drag



$$m\dot{v}_y = mg - bv_y \qquad (v_y > 0)$$

retarding force upward.

$$\boxed{\begin{aligned} &\text{when } mg - bv_y = 0 \\ &v_y = \frac{mg}{b} = v_{ter} \end{aligned}}$$

$$m\dot{v}_y = -b(v_y - v_{ter})$$

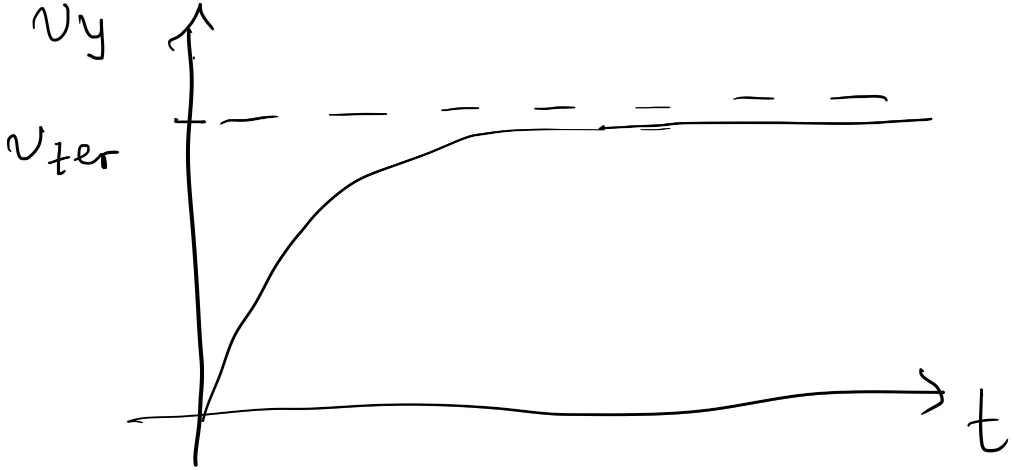
$$\frac{dv_y}{v_y - v_{ter}} = -\frac{b}{m}$$

$$\hookrightarrow \quad v_y - v_{ter} = A e^{-b/m\,t} \quad ; \quad t = 0,\ v_y = v_{y_0}$$

$$\boxed{A = v_{y_0} - v_{ter}}$$

$$\boxed{\begin{aligned} v_y(t) &= v_{ter} + (v_{y_0} - v_{ter})e^{-t/\tau} \\ &= v_{y_0}e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right) \end{aligned}}$$

$t \to \infty$, $v_y(t) \to v_{ter}$



Short time approx.

$v_y(t) \simeq$

$v_{y_0}(1 - t/\tau) + v_{ter}\, t/\tau$.

$\simeq v_{y_0} + (v_{ter} - v_{y_0})\, t/\tau$.

$v_{y_0} = 0$

$v_y(t) \simeq v_{ter}\, t/\tau$.

Next integration

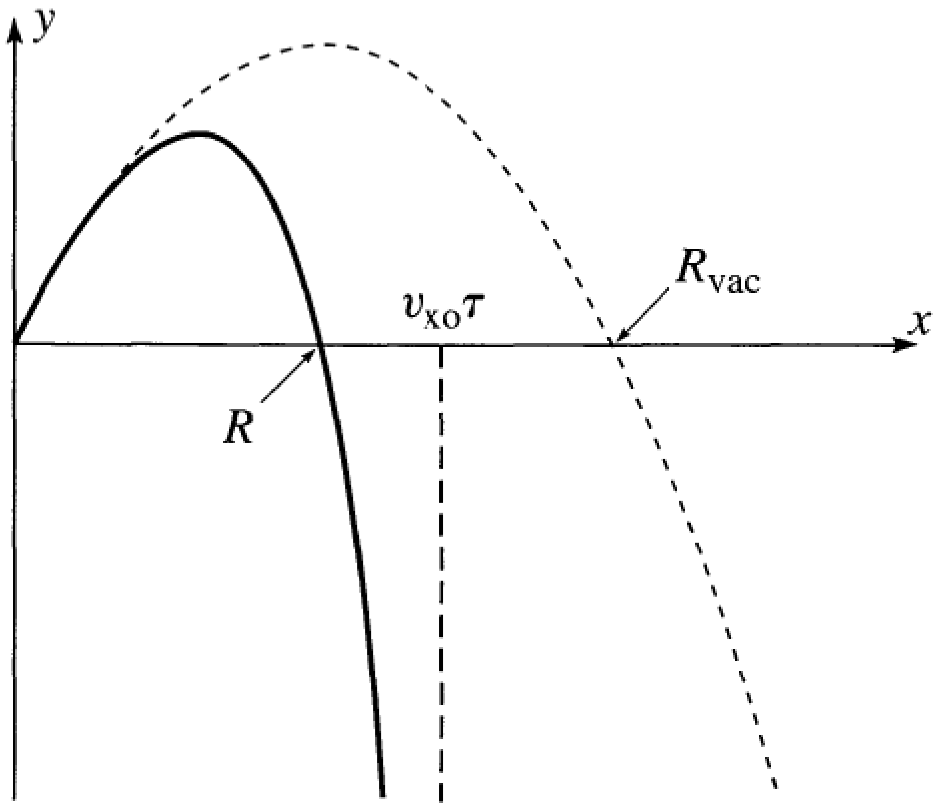
$$v_y(t) = v_{ter}\left(1 - e^{-t/\tau}\right)$$

$y(0) = 0$

$$\boxed{y(t) = v_{ter}\, t + (v_{y_0} - v_{ter})\,\tau\left(1 - e^{-t/\tau}\right)}$$

$$\boxed{x(t) = x_\infty\left(1 - e^{-t/\tau}\right) \qquad x_\infty = v_{x_0}\tau}$$

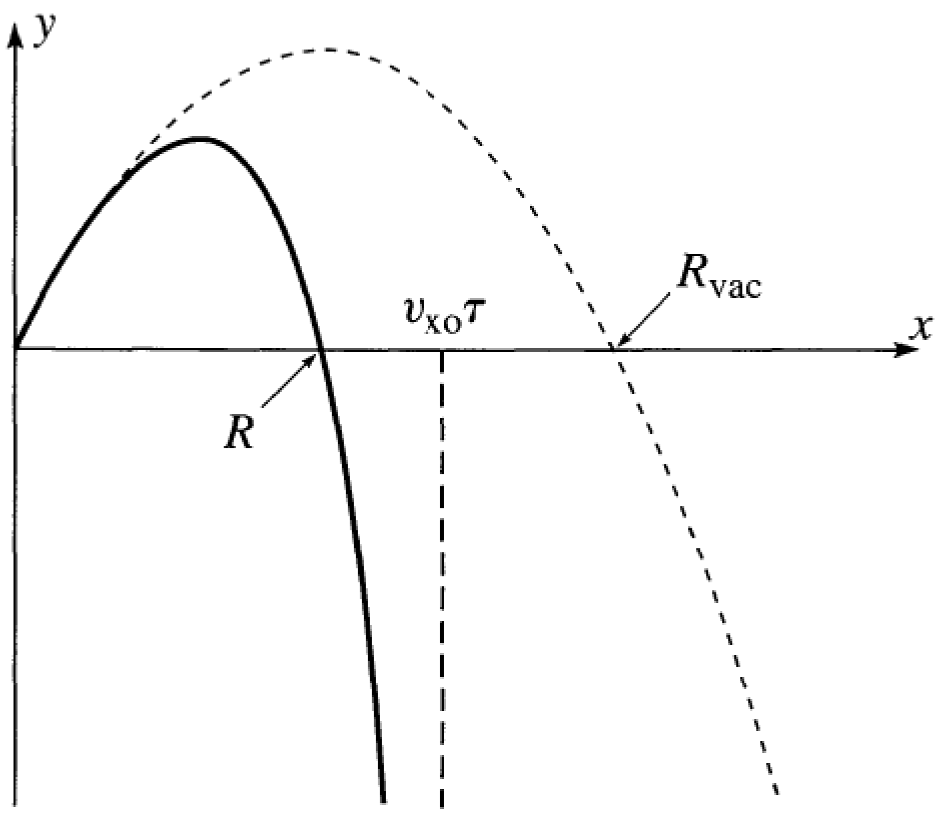
orbit of a projectile subject to linear drag.



y
x
$v_{xo}\tau$
R_{vac}
R

Physics I

Lecture 5



y
x
$v_{xo}\tau$
R_{vac}
R

Trajectory and range in a linear medium

slight change → will take upward y direction as +ve
↳ reverse the sign of v_{ter}

$$x(t) = v_{x_0}\tau\left(1 - e^{-t/\tau}\right) \quad \text{—(1)}$$

$$\boxed{\tau = \frac{m}{b}}$$

$$y(t) = \left(v_{y_0} + v_{ter}\right)\tau\left(1 - e^{-t/\tau}\right) - v_{ter}\,t \quad \text{—(2)}$$

eliminate t

$$\frac{x}{v_{x_0}\tau} = 1 - e^{-t/\tau}$$

$$\boxed{t = -\tau \ln\left(1 - \frac{x}{v_{x_0}\tau}\right)} \quad \text{—(3)}$$

Plug in 3 into 2

$$y = \frac{(v_{y_0} + v_{ter})x}{v_{x_0}} + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x_0}\tau}\right) \quad (4)$$

limit of small air resistance $\left\{\frac{1}{\tau} = \frac{b}{m}\right)\Big\}$ expand ln to lowest order

$\tau = m/b$

$$\boxed{y \simeq -\frac{1}{2} g \frac{x^2}{v_{x_0}^2}}$$ vacuum case.

$v_{ter}, \tau \to \infty$, vacuum case.

$x \to v_{x_0}\tau$, $y \to -\infty$ vertical asymptote

Horizontal Range

Recall $\boxed{R_{vac} = \frac{2 v_{x0} v_{y0}}{g}}$ case $b = 0$

Range: x for $y = 0$

$$y = \frac{(v_{y0} + v_{ter})x}{v_{x0}} + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$$0 = \frac{(v_{y0} + v_{ter})R}{v_{x0}} + v_{ter}\tau \ln\left(1 - \underbrace{\frac{R}{v_{x0}\tau}}_{\text{small for } b \text{ small}}\right)$$

$$\ln(1-\varepsilon) = -\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots\right).$$

large τ

$$\left(\frac{v_{y_0} + v_{ter}}{v_{x_0}}\right)R - v_{ter}\tau\left[\frac{R}{v_{x_0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x_0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x_0}\tau}\right)^3 + \cdots\right]$$

$$= 0$$

one trivial soln: $R = 0$

$$\boxed{\frac{v_{y_0}}{v_{x_0}} - \frac{v_{ter}R}{2\,v_{x_0}^2\tau} - \frac{1}{3}\frac{R^2}{v_{x_0}^3\tau^2} = 0}$$

$$\frac{v_{x0}}{v_{y0}} - \frac{v_{ter}}{2}\frac{R}{v_{x0}^2 \tau} - \frac{1}{3}\frac{R^2}{v_{x0}^2 \tau^2} v_{ter} = 0 .$$

$$R \cong \frac{2 v_{x0} v_{y0}}{g} - \underbrace{\frac{2}{3 v_{x0} \tau}}_{\to \text{ small}} R^2 \qquad \boxed{\frac{v_{ter}}{\tau} = g}$$

R at best R_{vac}.

first approx

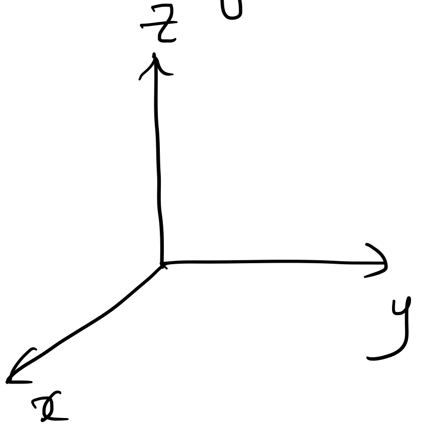
$$R \simeq \frac{2 v_{x0} v_{y0}}{g}$$

2nd approx

$$\boxed{R \simeq \frac{2 v_{x0} v_{y0}}{g} - \frac{2}{3 v_{x0} \tau} R_{vac}^2 \cong R_{vac}\left(1 - \frac{4}{3}\frac{v_{y0}}{v_{ter}}\right)}$$

Another example of a velocity dependent force.

Charge in a uniform magnetic field.



$$\vec{B} = B_0 \hat{y}$$

$$\vec{F}_{mag} = q\left(\vec{v} \times \vec{B}\right)$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

$$\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

Eqn. of motion $\vec{F} = m\vec{a}$

$$\boxed{\begin{aligned} m\ddot{x} &= (\vec{F}_{mag})_x = -qB_0\dot{z} \\ m\ddot{y} &= 0 \\ m\ddot{z} &= qB_0\dot{x} \end{aligned}}$$

$$y = \dot{y}_0 t + y_0 \quad (2)$$

$$y_0 = y(t=0)$$
$$\dot{y}_0 = \dot{y}(t=0)$$

$$\left.\begin{aligned} m\ddot{x} &= -qB_0\dot{z} \quad (1) \\ m\ddot{z} &= \cancel{-}\, qB_0\dot{x} \quad (3) \end{aligned}\right\}$$

$$\alpha = \frac{qB_0}{m}$$

$$\left.\begin{aligned} \ddot{x} &= -\alpha\dot{z} \\ \ddot{z} &= \alpha\dot{x} \end{aligned}\right\}$$

Take derivative

$$\left.\begin{aligned} \dddot{x} &= -\alpha\ddot{z} = -\alpha^2\dot{x} \\ \dddot{z} &= +\alpha\ddot{x} = -\alpha^2\dot{z} \end{aligned}\right\}$$

$\dot{z} = u \,, \quad \dot{x} = v$

$$\boxed{\ddot{u} = -\alpha^2 u \,, \quad \ddot{v} = -\alpha^2 v}$$

$\ddot{u} + \alpha^2 u = 0$

$\ddot{v} + \alpha^2 v = 0$

$\to \dot{z} = u = \tilde{A} \sin\alpha t + \tilde{B} \cos\alpha t$.

Integrate once

$$z = -\frac{\tilde{A}}{\alpha} \cos\alpha t + \frac{\tilde{B}}{\alpha} \sin\alpha t$$

$$\boxed{z(t) = A' \cos\alpha t + B' \sin\alpha t + z_0}$$

Similarly

$$\boxed{x = A \cos\alpha t + B \sin\alpha t + x_0}$$

Are A, B, A', B' all independent

$$\ddot{x} = -\alpha \dot{z}$$

$$\ddot{z} = \alpha \dot{x}$$

$$-\alpha^2 A\cos\alpha t - \alpha^2 B\sin\alpha t = -\alpha(-\alpha A'\sin\alpha t + \alpha B'\cos\alpha t)$$

valid for all t, $t = 0$, $t = \pi/2\alpha$.

$$-\alpha^2 A = -\alpha^2 B' \qquad -\alpha^2 B = \alpha^2 A'$$

$$A = B' \qquad B = A'$$

$$(x - x_0) = A\cos\alpha t + B\sin\alpha t$$

$$(y - y_0) = \dot{y}_0 t$$

$$(z - z_0) = -B\cos\alpha t + A\sin\alpha t .$$

$t = 0$ $\quad \dot{z} = \dot{z}_0$, and $\dot{x} = 0$.

$B = 0$, $\alpha A = \dot{z}_0$

$$\boxed{\begin{aligned} x - x_0 &= \frac{\dot{z}_0}{\alpha}\cos\alpha t \\ (y - y_0) &= \dot{y}_0 t \\ (z - z_0) &= \frac{\dot{z}_0}{\alpha}\sin\alpha t \end{aligned}}$$

what trajectory is this ?.

$$(x - x_0)^2 + (z - z_0)^2 = \left(\frac{\dot{z}_0^{\,2} m^2}{q^2 B_0^2}\right)$$

right circular helix .

Physics I

Lecture 6

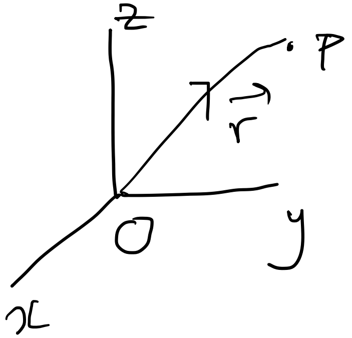
Conservation Laws

1. Linear momentum

$$\dot{\vec{p}} = \vec{F}, \quad \vec{F} = 0, \quad \dot{\vec{p}} = 0$$

$$\boxed{\vec{p} = \text{const}}, \text{ when } \vec{F} = 0$$

2. Angular momentum



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \underbrace{\dot{\vec{r}} \times \vec{p}}_{=0} + \underbrace{\vec{r} \times \dot{\vec{p}}}_{\vec{r} \times \vec{F}}$$

$$= \vec{r} \times \vec{F} = \vec{N} \Rightarrow \text{torque}$$

$$\boxed{\vec{L} = \text{const}} \text{ when } \vec{N} = 0$$

3. Work

Work done on a particle by force $\vec{F}$ in taking it from config 1 to config 2.

$$\boxed{W_{12} = \int_1^2 \vec{F}\cdot d\vec{r}}$$

$$\vec{F}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot d\vec{r} = m\frac{d\vec{v}}{dt}\cdot\frac{d\vec{r}}{dt}\,dt$$

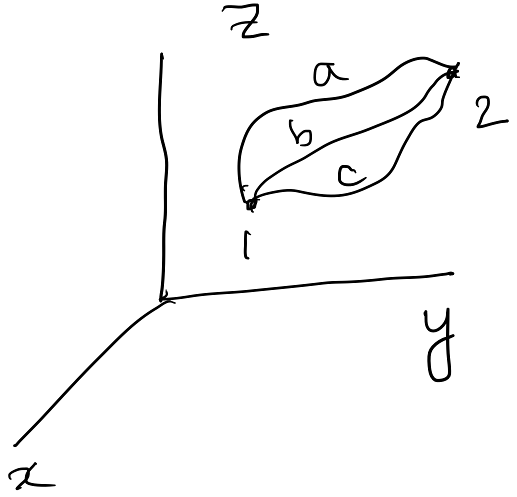
$$= m\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}m\frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt$$

$$\vec{F}\cdot d\vec{r} = d\left(\frac{1}{2}mv^2\right).$$

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 d\left(\frac{1}{2}mv^2\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\boxed{W_{12} = T_2 - T_1} = \Delta T \, .$$ Work-energy theorem



In general

$\int_1^2 \vec{F}\cdot d\vec{r}$ depends on path

But there is a class of forces for which work done does not depend on path. $\Longrightarrow$ Conservative forces.

Conditions for a force to be conservative

(i) $\vec{F}$ depends only on position $\vec{r}$ (not on velocity or time)

$\vec{F} = \vec{F}(\vec{r})$.

(ii) For any two points 1 & 2, $W(1 \to 2)$ done by $\vec{F}$ must be independent of path.

$\hookrightarrow$ Possible to define a quantity U, called potential energy $U(\vec{r})$

$$\boxed{U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'}$$

standard position

$$\int_1^2 \vec{F}(\vec{r}\,')\cdot d\vec{r}\,' = \int_1^{\vec{r}_0} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,' + \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

$$= U_1 - U_2 = -\Delta U = -(U_2 - U_1)$$

But recall

$$-(U_2 - U_1) = T_2 - T_1 .$$

$$\boxed{T_1 + U_1 = T_2 + U_2 = E = \text{const}}$$

→ Total mechanical energy = const for conservative forces.

Non conservative forces

$$\vec{F} = \vec{F}_{cons} + \vec{F}_{NC}.$$

$$\Delta T = W = W_{cons} + W_{nc}.$$

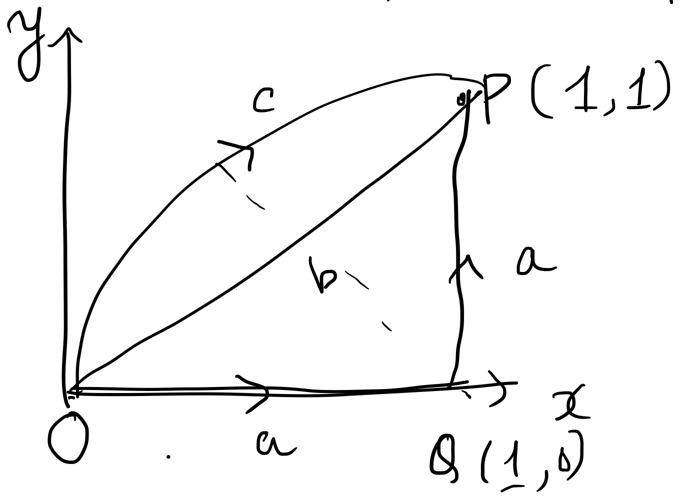
$$= -\Delta U + W_{nc}.$$

$$\boxed{\Delta E = \Delta(T+U) = W_{nc}.}$$

2 , $\underbrace{y + 2x}$. cannot happen.

1 Digression

example of path dependence of a line integral



$$\vec{F} = y\hat{x} + 2x\hat{y}$$

$$W_a = \int_a \vec{F}\cdot d\vec{r}$$

$$= \int_O^Q \vec{F}\cdot d\vec{r} + \int_Q^P \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 F_x\, dx + \int_0^1 F_y\, dy .$$

$$= \int_0^1 0\, dx + \int_0^1 2\, dy .$$

$$= 2 .$$

$$W_b = \int_b F_x\, dx + \int_b F_y\, dy .$$

$$= 3/2 .$$

Force as a gradient of potential energy

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

suggests that $\vec{F}(\vec{r})$ can be written as some kind of ~~denial~~ derivative of $U(\vec{r})$. In 1 d you know

$$F(x) = -\frac{dU}{dx}.$$

$$W(\vec{r} \to \vec{r}+d\vec{r}) = \vec{F}(\vec{r})\cdot d\vec{r} = -dU$$

$$= -\left[U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)\right].$$

In 1d $\quad df = \frac{df}{dx}\,dx.$

$$dU = U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)$$

$$= \left[\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right]$$

$$W(\vec{r} \to \vec{r} + d\vec{r}) = -\left[\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right]$$

$$= F_x dx + F_y dy + F_z dz .$$

$$\Rightarrow \quad F_x = -\frac{\partial U}{\partial x} \ ; \quad F_y = -\frac{\partial U}{\partial y} \ ; \quad F_z = -\frac{\partial U}{\partial z}$$

$$\boxed{\vec{F} = -\hat{x}\frac{\partial U}{\partial x} - \hat{y}\frac{\partial U}{\partial y} - \hat{z}\frac{\partial U}{\partial z}}$$

Gradient :

given any scalar fn $\phi(x,y,z)$.

$$\vec{\nabla}\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

$\vec{\nabla}$ → vector operator

$$\boxed{\vec{F} = -\vec{\nabla} U}$$

$\vec{\nabla} U \cdot d\vec{r} = dU$ ⟹ check.

Physics I

Lecture 7

Conditions for a Force to be Conservative

A force **F** acting on a particle is **conservative** if and only if it satisfies two conditions:

(i) **F** depends only on the particle's position **r** (and not on the velocity **v**, or the time t, or any other variable); that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$.

(ii) For any two points 1 and 2, the work $W(1 \to 2)$ done by **F** is the same for all paths between 1 and 2.

$$U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

Total Mechanical energy

$E = T + U$ conserved

$$\vec{F} = -\vec{\nabla} U$$ "Any conservative force is derivable from a potential energy"

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

↳ vector operator

$$\vec{\nabla} U = \hat{x}\frac{\partial U}{\partial x} + \hat{y}\frac{\partial U}{\partial y} + \hat{z}\frac{\partial U}{\partial z}$$

$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

$$F_x = -\frac{\partial U}{\partial x} \quad ; \quad F_y = -\frac{\partial U}{\partial y} \quad ; \quad F_z = -\frac{\partial U}{\partial z}$$

example: $U = Axy^2 + B\sin Cz$ A, B, C are constants

$$\vec{F} = -\vec{\nabla} U$$

$$F_x = -\frac{\partial U}{\partial x} = -Ay^2$$

$$F_y = -\frac{\partial U}{\partial y} = -2Axy$$

$$F_z = -\frac{\partial U}{\partial z} = -CB\cos Cz$$

2nd Condition for $\vec{F}$ to be conservative

Recall for $\vec{F}$ to be conservative

$$W(\vec{r}_0 - \vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}' \longrightarrow \text{path independent}$$

Can we find a differential equivalent criterion to test whether a force is conservative?

Yes.

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

curl of a vector, Given $\vec{A}$, $\boxed{\text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A}}$

It can be shown that $\int_1^2 \vec{F} \cdot d\vec{r}$ is independent of path iff

$$\boxed{\vec{\nabla} \times \vec{F} = 0}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right).$$

If $\vec{F} = -\vec{\nabla} U$, $\vec{\nabla}\times\vec{F} = 0$

[Ex : $\vec{\nabla}\times\vec{\nabla}\phi = 0$ show : identity].

Coulomb Force

$$\vec{F} = \frac{kqQ}{r^2}\hat{r} = \frac{\alpha}{r^3}\vec{r} = \frac{\alpha}{r^3}(x\hat{x} + y\hat{y} + z\hat{z}).$$

$$\vec{\nabla}\times\vec{F} = 0 \;?$$

$$F_x = \frac{\alpha x}{r^3}, \quad F_y = \frac{\alpha y}{r^3}, \quad F_z = \frac{\alpha z}{r^3}.$$

$$(\vec{\nabla}\times\vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}.$$

$$= \frac{\partial}{\partial y}\left(\frac{\alpha z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{\alpha y}{r^3}\right).$$

$$= -(3\alpha yz)/r^5 + \frac{3\alpha yz}{r^5} = 0$$

$$(\vec{\nabla}\times\vec{F})_x = (\vec{\nabla}\times\vec{F})_y = (\vec{\nabla}\times\vec{F})_z$$

$$\boxed{\vec{\nabla}\times\vec{F} = 0} \qquad \vec{F} \to \text{conservative}.$$

Potential energy exists.

$$U(\vec{r}) = \frac{\alpha}{r} \qquad r_0 \to \infty \qquad U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r})\cdot d\vec{r}.$$

$$U(\vec{r}) = \frac{kqQ}{r} \qquad \vec{F} = -\vec{\nabla}U.$$

$$(\vec{\nabla}U)_x = \frac{\partial}{\partial x}\left(\frac{kQq}{r}\right) = \frac{-kqQ}{r^2}\frac{\partial r}{\partial x} = \frac{-kQqx}{r^3}$$

$$\left\{ \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \right\}$$

$$(\vec{\nabla} U)_y = \frac{-kqQy}{r^3} = F_y \quad \text{and so on}$$

$$\vec{F} = -\vec{\nabla} U = \frac{kqQ\vec{r}}{r^3} = \frac{kQq}{r^2}\hat{r} .$$

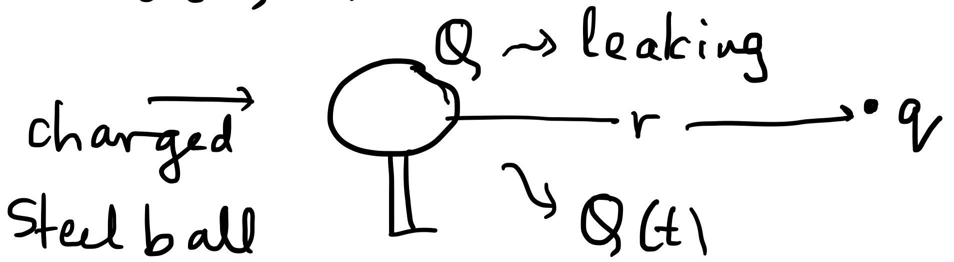
Time dependent potential energy

$\vec{F}(\vec{r}, t)$ say $\vec{F}$ satisfies $\vec{\nabla} \times \vec{F} = 0$

but not 1st condition.

non-conservative, but can still define

$U(\vec{r}, t)$ $\quad \vec{F} = -\vec{\nabla} U$



$$\vec{F} = \frac{k\,Q(t)\,q}{r^2}\hat{r}$$

$$\vec{F} = \frac{kqQ(t)}{r^2}\,\hat{r}$$

$$\vec{\nabla} \times \vec{F} = 0$$

$$\vec{F} = -\vec{\nabla}\, U(\vec{r}, t)$$

$$E = T + U \quad ; \quad dT = \frac{dT}{dt}\, dt = (m\dot{\vec{v}} \cdot \vec{v})\, dt$$
$$= \vec{F} \cdot d\vec{r}$$

$$dU = \underbrace{\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz}_{} + \frac{\partial U}{\partial t} dt .$$

$$dU = \vec{\nabla} U \cdot d\vec{r} + \left(\frac{\partial U}{\partial t}\right) dt$$

$$dU = \vec{\nabla} U \cdot d\vec{r} + \frac{\partial U}{\partial t}\, dt$$

$$= -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t}\, dt$$

$$dE = d(T + U)$$

$$= dT + dU$$

$$= \cancel{\vec{F} \cdot d\vec{r}} - \cancel{\vec{F} \cdot d\vec{r}} + \frac{\partial U}{\partial t}\, dt$$

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} = 0 \quad \text{only when } \frac{\partial U}{\partial t} = 0$$

$E = U + T$ conserved

U does not explicitly depend on time

Physics I

Lecture 8

One dimensional linear motion

- $F = F(x)$: satisfies the 1st condition for conservative force

↳ 2nd condition $W_{12} = \int_1^2 F(x)\,dx$ is path independent

↳ Is automatically satisfied in 1D.

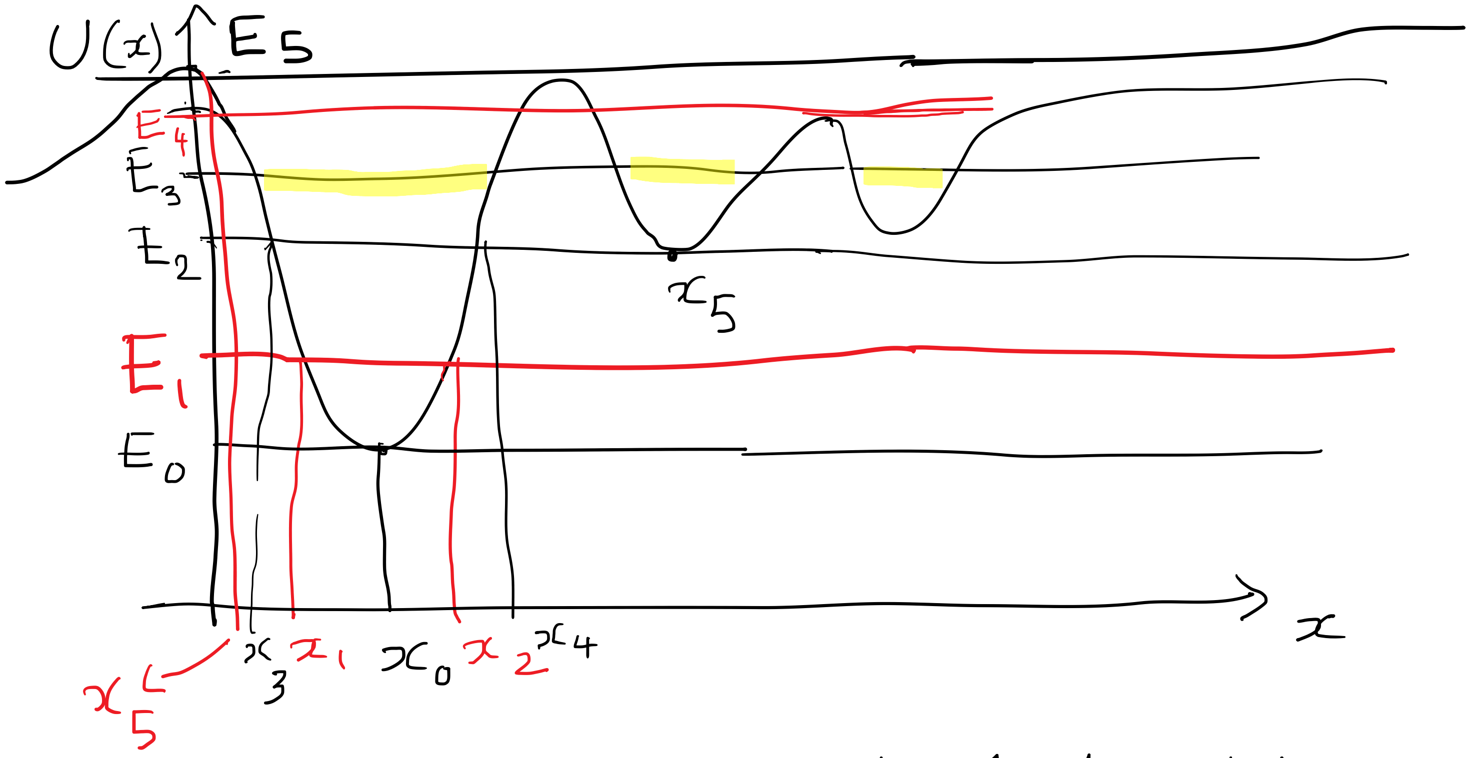
$$W_{ABCB} = W_{AB} + \underbrace{W_{BC} + W_{CB}}_{=0}$$

If $F = F(x) \longrightarrow U(x)$ exists $\quad E = T + U = \text{const}$.

- We can learn great deal about the motion by looking at graph of $U(x)$ without explicitly obtaining solution.

- Obs. 1 $T \geqslant 0$. $\therefore$ for a given energy E, the motion will be confined to regions of the x-axis where

$\{E = T + U\}$ $\boxed{U(x) \leqslant E}$ $\longrightarrow$ classically allowed region.



[$E < E_0 \Rightarrow T < 0$
in response to question in class]

{ $E = E_0 = T + U$
$= T + U_0$
$T = 0$ }

- With energy E_0, particle will just sit at x_0
- With energy E_1, classically allowed region $x_1 \leq x \leq x_2$
  x_1, x_2 called turning points, $\boxed{U(x) = E_1}$, the motion must be bounded, oscillatory.
- With energy E_2, classically allowed regions are $x_3 \leq x \leq x_4$
  $x = x_5$; either oscillate between x_3 & x_4 turning pts
  or sit at x_5

- With energy E_5, there is only <u>one</u> turning point, particle comes in from ∞ hits barrier/turning pt and goes back to ∞ along x-axis, speeding up over the valleys and slowing down ~~over the~~ at the hill. Unbounded motion.

- With energy $> E_5$ no turning points and particle moves in one direction only modulating the speed according to the depth of the potential

x_0 : Stable equilibrium position

$$U(x) = \underbrace{U(x_0)}_{\text{redefine ref pt.}} + \underbrace{\left(\frac{dU}{dx}\right)_{x_0}}_{0} (x - x_0) + \frac{1}{2}\left(\frac{d^2U}{dx^2}\right)_{x_0} (x-x_0)^2 + \cdots$$

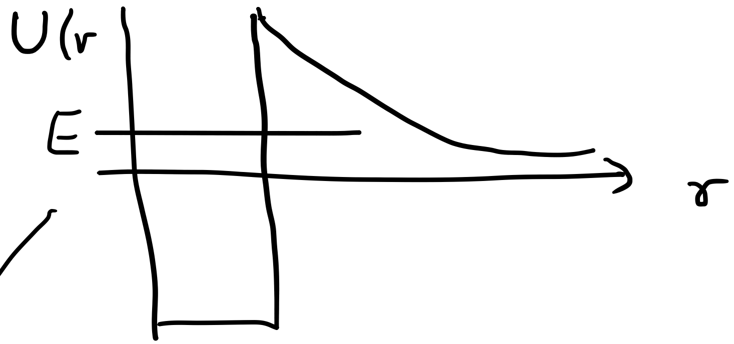
$$U(x) \simeq \frac{1}{2} k (x - x_0)^2$$

$$\left(\frac{dU}{dx}\right)_{x_0} = 0 \quad , \quad \left(\frac{d^2U}{dx^2}\right)_{x_0} \geqslant 0 \Rightarrow \text{stable}.$$

$$\left(\frac{dU}{dx^2}\right)_{x_0} \leqslant 0 \Rightarrow \text{unstable}.$$

Why "classically" allowed ?

α-decay



energy of α-particles

nuclear potential.

in quantum mech.

$U(x) \leq E$ condn. violated.

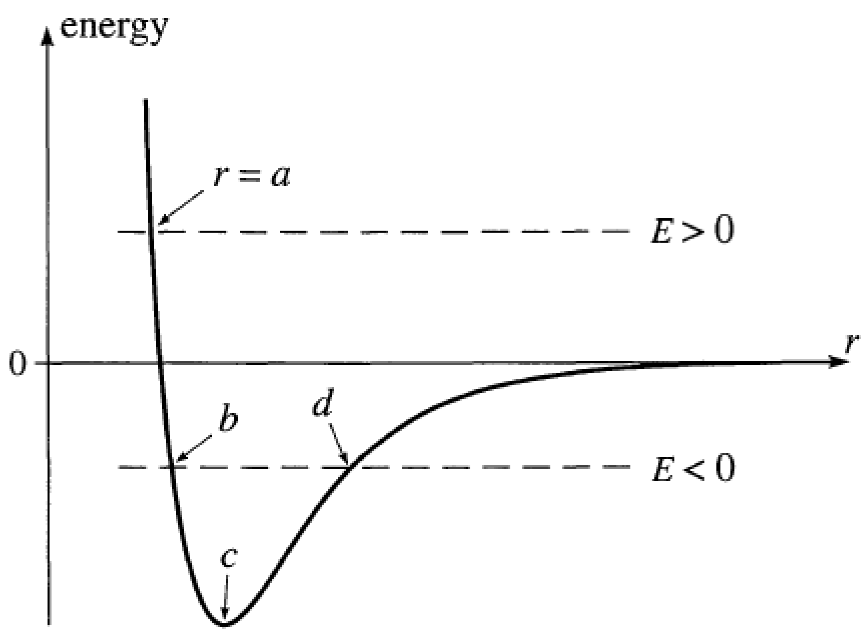


Figure 4.12 The potential energy for a typical diatomic molecule such as HCl, plotted as a function of the distance r between the two atoms. If $E > 0$, the two atoms cannot approach closer than the turning point $r = a$, but they can move apart to infinity. If $E < 0$, they are trapped between the turning points at b and d and form a bound molecule. The equilibrium separation is $r = c$.

- One-dimension motion can be completely solved in principle.

$$E = \frac{1}{2} m v^2 + U(x)$$

$$v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

$$x(t) = \; ?$$

integrate

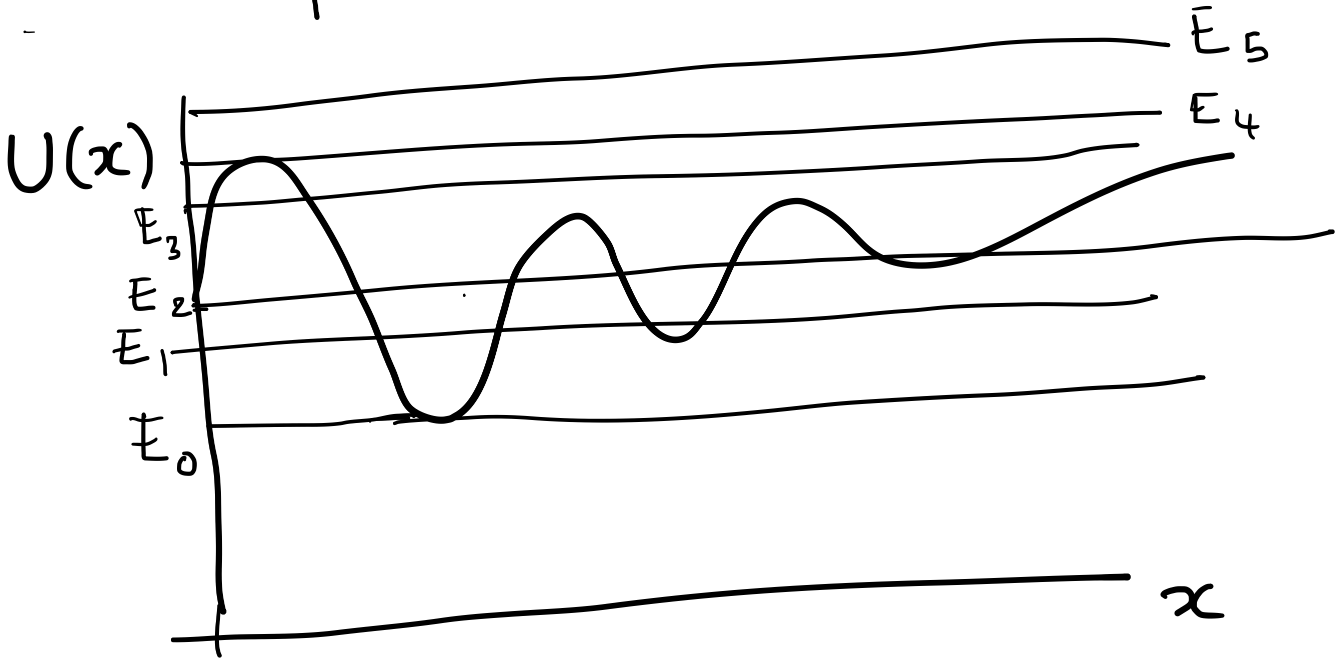
$$\boxed{t - t_0 = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \left[E - U(x')\right]^{-1/2} dx'} \longrightarrow \text{complete soln.}$$

two initial cond E, x_0

Physics I

Lecture 9

Recap: 1 d linear motion



- **Bounded vs unbounded**
- **turning points**
- **classically allowed regions.** $U(x) \leq E$
- **stable vs unstable equilibrium**

Completely solvable system

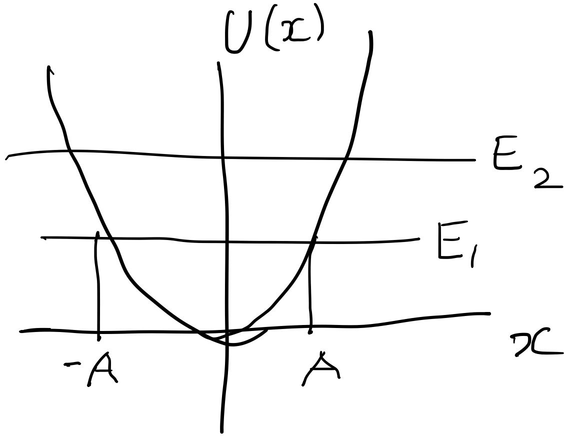
$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

can be formally integrated $\longrightarrow$ $$t - t_0 = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \left[E - U(x')\right]^{-1/2} dx'$$

Oscillations

Simple harmonic oscillator

$$F = -kx \, , \quad U(x) = \frac{1}{2} k x^2$$



$\longrightarrow$ for all E motion is bounded and oscillatory

Turning points are symmetric about the origin

Equation of motion

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

S.H.O is an exactly solvable problem

Any potential can be approximated as S.H.M close to a minimum of potential

$$U(x) \simeq \underbrace{U(x_0)}_{\substack{\text{can be}\\ \text{set to zero}}} + \underbrace{\left.\frac{dU}{dx}\right|_{x=x_0}}_{=0} (x-x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} (x-x_0)^2 + \cdots$$

$$\simeq \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0} (x-x_0)^2 .$$

Linear ordinary differential eqns with constant coefficients

no higher than 1st degree in dependent variable and derivatives.

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t)\frac{dx}{dt} + a_0(t) = b(t).$$

order n

$b(t) = 0 \implies$ homogeneous

We will mostly be concerned with 2nd order.

General soln of any 2nd order eqn will contain 2 arbitrary constants

$$x = x(t; c_1, c_2)$$

Theorem 1 : If $x = x(t)$ is a soln of ~~any~~ a linear homogeneous differential eqn, then $x_1 = C x_1(t)$ is also a soln where C is a const.

Theorem 2 : If $x = x_1(t)$ and $x = x_2(t)$ are solutions of a linear homogeneous differential eqn, then $\boxed{x = x_1(t) + x_2(t)}$

is also a soln.

2nd order.

General soln is given by $C_1 x_1(t) + C_2 x_2(t)$ where x_1 and x_2 are linearly independent solns and C_1 and C_2 are real constants.

$x_1(t)$ and $x_2(t)$ are said to be linearly independent iff

$$\lambda x_1(t) + \mu x_2(t) = 0 \quad \text{only for } \begin{matrix} \lambda = 0 \\ \mu = 0 \end{matrix}$$

Consider 2nd order diff eqns with const coeff.

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad \text{--- (1)}$$

Assume a soln of form $x = e^{pt}$, {Ansatz}.

plug ansatz into (1)

$$\boxed{a_2 p^2 + a_1 p + a_0 = 0} \Rightarrow \text{auxilliary eqn} \quad \text{--- (2)}$$

$$p^2 + ap + b = 0 \qquad a = \frac{a_1}{a_2}, \; b = \frac{a_0}{a_2}$$

$$p = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

p_1, p_2 ; If $p_1 = p_2$, 1 soln.

for $p_1 \neq p_2$

$$\boxed{x = c_1 e^{p_1 t} + c_2 e^{p_2 t}}$$

If $p = p_1 = p_2$, only one soln by this method, Verify that te^{pt} is also a soln and is linearly independent.

Harmonic Oscillator

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

$x \sim e^{pt} \longrightarrow p^2 + \omega_0^2 = 0$ auxillary eqn.

$$p = \pm i\omega_0$$

$x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$, In general C_1, C_2 can be complex

but x must be real so must restrict C_1, C_2 accordingly

$C = C_2 = C_1^*$ will ensure a real soln.

$$x = C e^{i\omega_0 t} + C^* e^{-i\omega_0 t} . \qquad C = \frac{1}{2} A e^{-i\delta}$$

$$\boxed{x = A \cos(\omega_0 t - \delta) .}$$

2 arbitrary constants.

equivalently written

$$x = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t$$

or

$$x = A \cancel{\cos} \sin(\omega_0 t - \phi)$$

Another comment

$$\ddot{x} + \omega_0^2 x = 0$$

↳ since contains only real coeff → soln. will be real.

A complex fn. can satisfy this diff, its real and imaginary parts satisfy ~~to~~ it separately.

say $x = C e^{i\omega_0 t} = A e^{i(\omega_0 t - \delta)} = A\cos(\omega_0 t - \delta) + i A \sin(\omega_0 t - \delta)$.

Physics I

Lecture 10

Harmonic oscillator in 1D.

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_0^2 x = 0$$

$$\rightarrow \quad x = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\text{or} \quad \tilde{A}\sin(\omega_0 t - \delta).$$

Harmonic oscillator in 2D

$$\vec{F} = -k\vec{r} \qquad F_x = -kx, \quad F_y = -ky$$

$$\left.\begin{aligned} x(t) &= A\cos(\omega_0 t - \alpha) \\ y(t) &= B\cos(\omega_0 t - \beta) \end{aligned}\right\}$$

Please read this section in book.

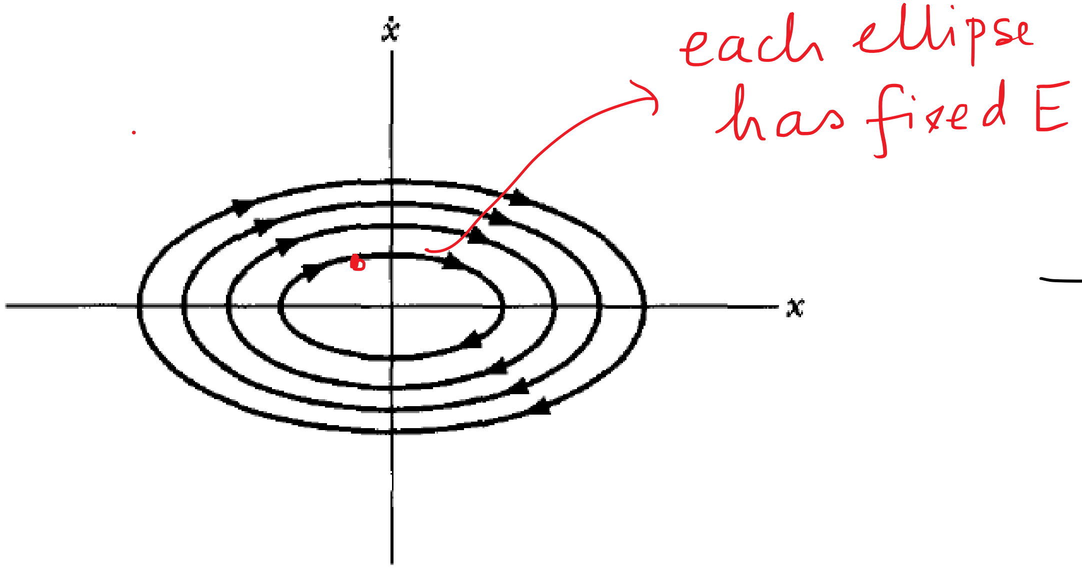
Phase diagrams

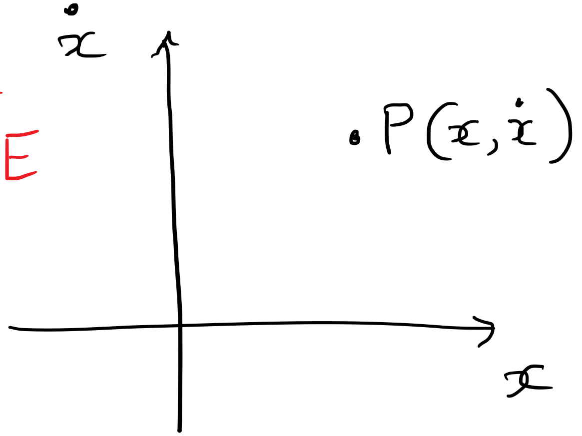
$$F = -kx$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\boxed{E = \frac{1}{2}kA^2}$$

const.





For any dynamical system, specifying $x, \dot{x}$

$x(t_0), \dot{x}(t_0)$ complete specification of state

$(x, \dot{x}) \longrightarrow$ phase space, In the case of 1-d motion phase space is 2D.

[gas: N particles in 3D

dimension of phase space $\equiv 6N$]

$x = A\sin(\omega_0 t - \delta)$

$\dot{x} = A\omega_0 \cos(\omega_0 t - \delta)$.

eliminate t

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1 \longrightarrow \text{ellipse in phase space.}$$

Recall $E \propto A^2$

Two phase trajectories can never cross.

clockwise trajectories, $x > 0$, $\dot{x}$ decreasing

$x < 0$ $\dot{x}$ increasing

Damped oscillations

Undamped case

$$\left.\begin{aligned} m\ddot{x} + kx &= 0 \\ \ddot{x} + \omega_0^2 x &= 0 \end{aligned}\right\}$$

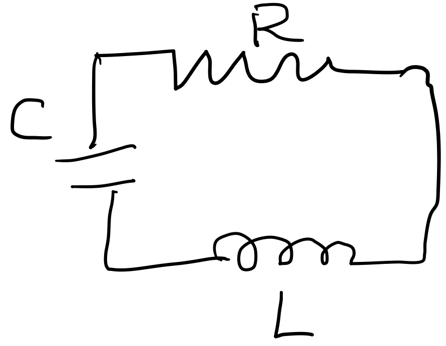
Add damping

$$m\ddot{x} = -kx - b\dot{x}$$

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

Interesting analogy

LCR circuit



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

$q \to x$

$L \to m$

$R \to b$

$\frac{1}{C} \to k$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$2\beta = \frac{b}{m}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 = \frac{k}{m}.$$

Let $x = e^{pt}$

$$p^2 + 2\beta p + \omega_0^2 = 0 \implies \text{auxilliary eqn.}$$

two solns

$$\left.\begin{aligned} p_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ p_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}\right\} \quad e^{p_1 t},\ e^{p_2 t} \text{ solns.}$$

General soln:

$$x(t) = C_1 e^{P_1 t} + C_2 e^{P_2 t}.$$

$$\boxed{x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)}$$

Cases :

1. undamped, $\beta = 0$, $x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$
2. underdamped $\beta^2 < \omega_0^2$
3. critically damped $\beta = \omega_0$
4. Overdamped $\beta^2 > \omega_0^2$.

Underdamped

Define $\sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\omega_1$

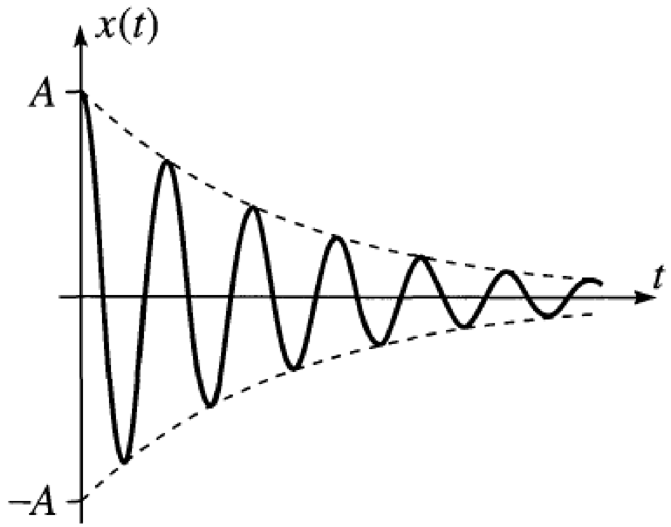
$$x(t) = e^{-\beta t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$$

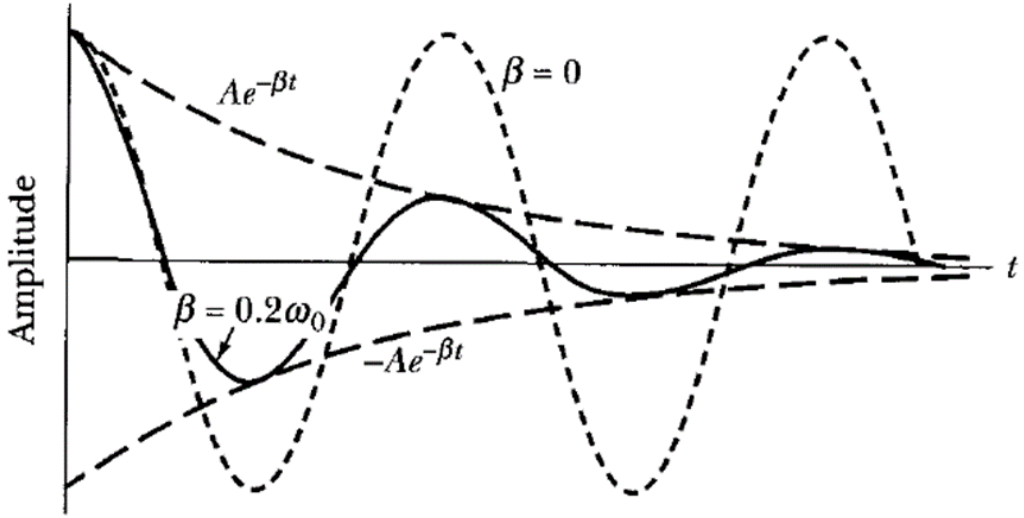
writing $C_1 = \frac{A}{2}e^{-i\delta}$, $C_2 = \frac{A}{2}e^{+i\delta}$

$$\boxed{x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)}$$

decaying amplitude.

β has dimensions of t^{-1}, $\frac{1}{\beta}$ is the time in which the amplitude falls to $\frac{1}{e}A$





underdamped case.

$$x_{env} = \pm A\, e^{-\beta t}.$$

notice that $\omega_1 < \omega_0$.

Undamped $E = \frac{1}{2}kA^2 \longrightarrow$ conserved.

$\beta \ll \omega_0$ $\quad \omega_1 \simeq \omega_0$

$$E \simeq \frac{1}{2}kA^2 e^{-2\beta t}$$

Critically damped

$\beta = \omega_0$ coincident roots. one soln. $x = e^{-\beta t}$

Another linearly ind. soln $x = t e^{-\beta t}$

$$x(t) = c_1 e^{-\beta t} + c_2 t e^{-\beta t}$$

$$\boxed{x(t) = e^{-\beta t}(c_1 + c_2 t)}$$

decay parameter $= \beta$

Overdamped case

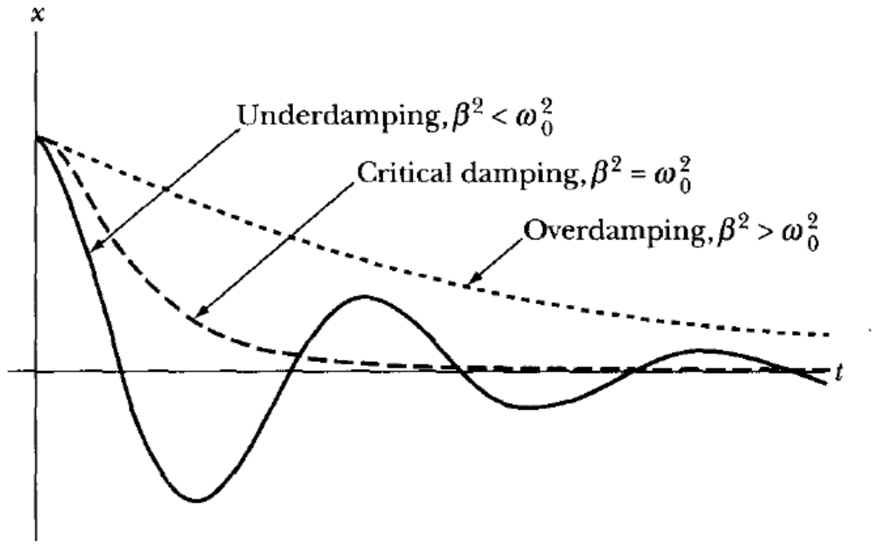
$\beta > \omega_0$ sq. root is real

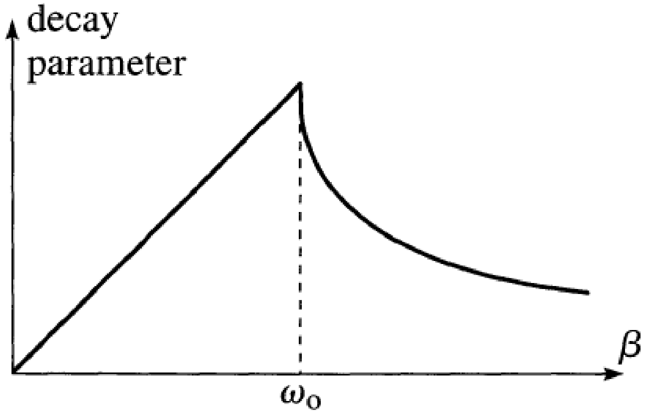
$$x(t) = c_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + c_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}.$$

exponential decay.

first term decays slower, dominates.

decay parameter $\beta - \sqrt{\beta^2 - \omega_0^2}$.





decay parameter

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$

Physics I

Lecture 11

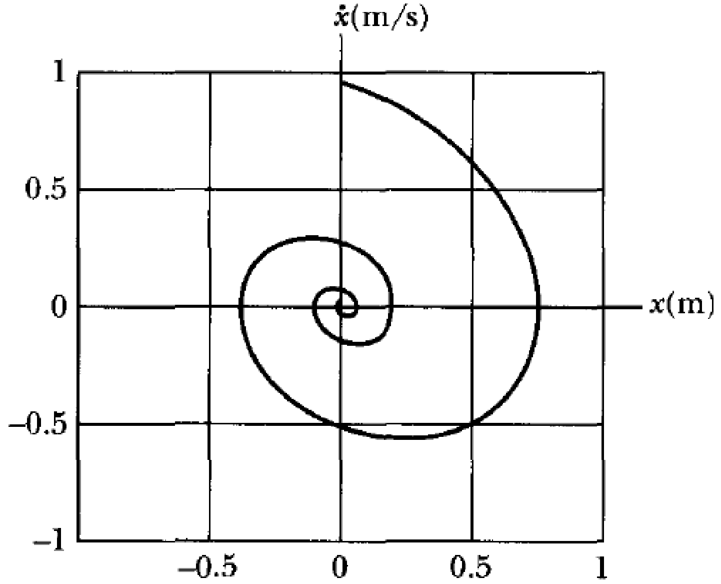
Damped harmonic oscillator (recap)

$$\ddot{x} + 2\beta x + \omega_0^2 x = 0$$

- Underdamped $\beta < \omega_0$
- Critically damped $\beta = \omega_0$
- overdamped $\beta > \omega_0$

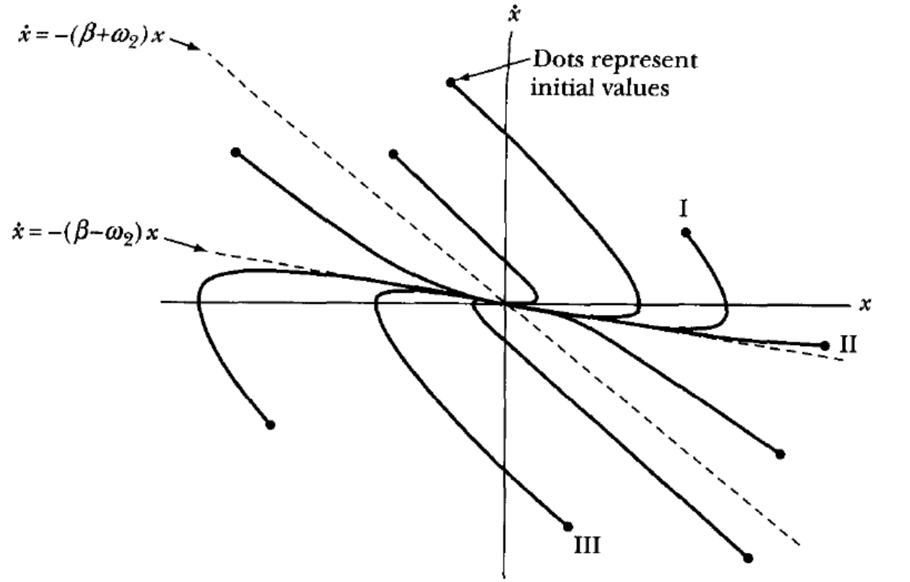
PHASE SPACE PLOTS

$\omega_2 = \sqrt{\beta^2 - \omega_0^2}$



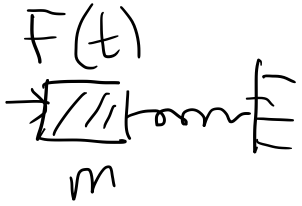
(b)

underdamped



overdamped

Forced/Driven Damped harmonic oscillator



$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad f(t) = \frac{F}{m}.$$

$\hookrightarrow$ inhomogeneous diff eqn.

<u>Theorem</u> : If $x_p(t)$ (particular soln.) is a solution of an inhomogeneous diff. eqn. and $x_h(t)$ is a soln to the corresponding homogeneous eqn., then $x_p(t) + x_h(t)$ is also a soln. to the inhomogeneous eqn..

$$\boxed{\text{General soln.}: \quad x_h(t) + x_p(t)}$$

We will specialize to ω ; driving frequency.

$$f(t) = f_0 \cos\omega t$$

$$\boxed{\frac{F(t)}{m} = f(t)}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t \rightsquigarrow \mathrm{Re}\left(f_0 e^{i\omega t}\right).$$

Assume a complex soln of the form $z = Ce^{i\omega t}$ — (1)

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} \quad \text{— (2)}$$

Plug (1) into (2)

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) Ce^{i\omega t} = f_0 e^{i\omega t}$$

$z = Ce^{i\omega t}$ is a soln, provided

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = Ae^{-i\delta} \quad A, \delta \text{ real}.$$

$$\boxed{A^2 = CC^* = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} \quad (3)$$

$f_0 e^{i\delta} = A(\omega_0^2 - \omega^2 + 2i\beta\omega)$

check that

$$\boxed{\delta = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}} \quad (4)$$

Soln:

$$\boxed{z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}} \quad (4)$$

Homogeneous soln:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x \sim e^{pt}$$

$$x_h = C_1 e^{p_1 t} + C_2 e^{p_2 t} \quad (5)$$

$$\boxed{p_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}}$$

dies out at late time
transient

Full soln. $x_p(t) + x_h(t)$

Let us specialize to underdamped case

$$\boxed{x(t) = A\cos(\omega t - \delta) + A_{tr}e^{-\beta t}\cos(\omega_1 t - \delta_{tr})} \quad (6)$$

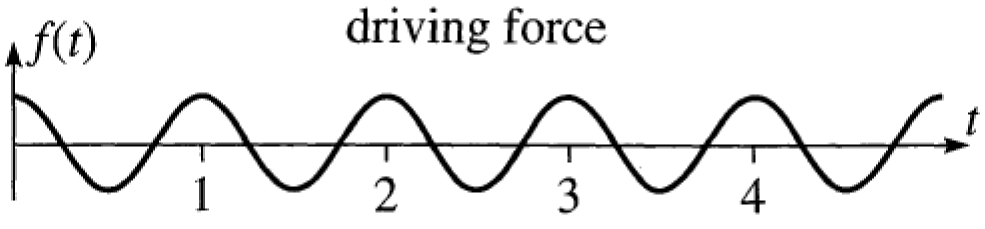
Arbitrary const to be fixed by initial conditions

{ Recall underdamped soln.

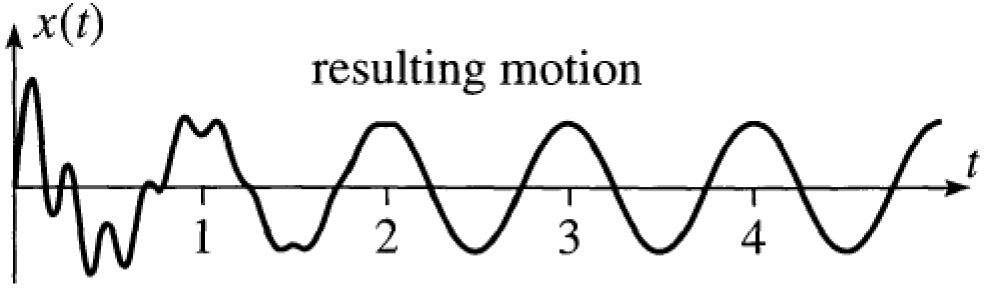
$$x(t) = Ce^{-\beta t}\cos(\omega_1 t - \phi) \}$$

$C \equiv A_{tr}$ $\quad$ tr : transient
$\phi \equiv \delta_{tr}$ }

wipes out memory of initial conditions.



(a)



(b)

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

- $A \propto f_0$
- phase lag between the driving force and resulting motion

driving force $\rightarrow f_0 \cos(\omega t)$

resulting motion $\rightarrow A\cos(\omega t - \delta)$

Resonance

$$A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

Maximum Amplitude

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = 0 \quad \rightsquigarrow \quad \boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

Res. freq is lowered as β increases

No res. will occur for $\beta > \frac{\omega_0}{2}$.

1. free oscillations, no damping

$$\omega_0^2 = \frac{k}{m}$$

2. + damping

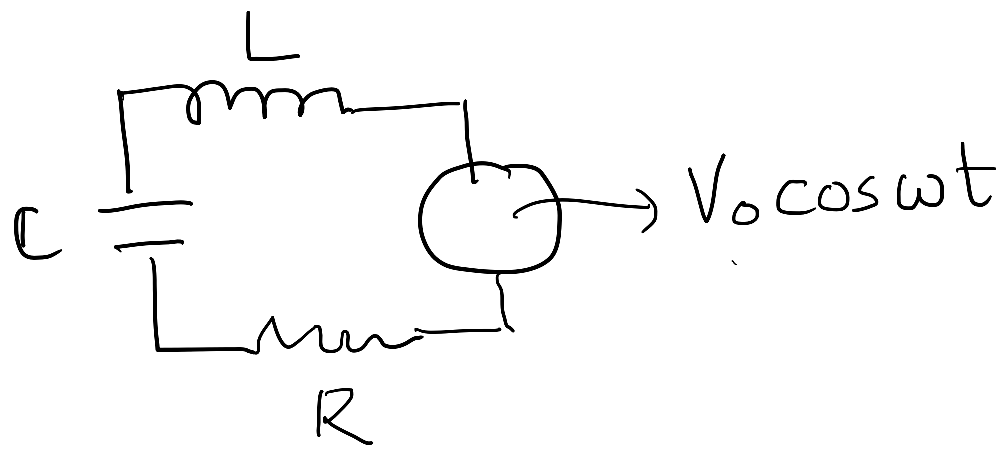
$$\omega_1^2 = \omega_0^2 - \beta^2$$

3. + Driving force.

$$\omega_R^2 = \omega_0^2 - 2\beta^2$$

$$\omega_0 > \omega_1 > \omega_R .$$

Analog LCR



$$m \equiv L$$

$$k \equiv \frac{1}{C}$$

$$R = 2\beta\,.$$

$$\frac{V_0}{L} \equiv f_0$$

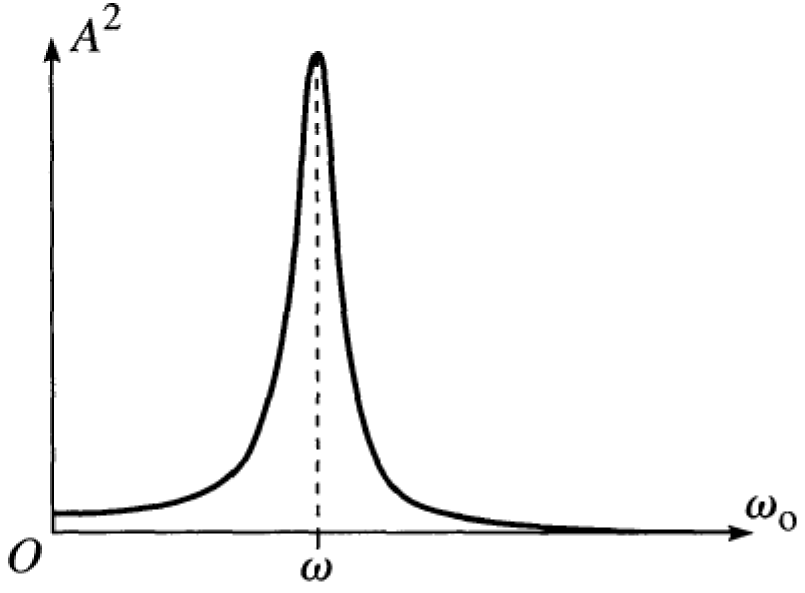
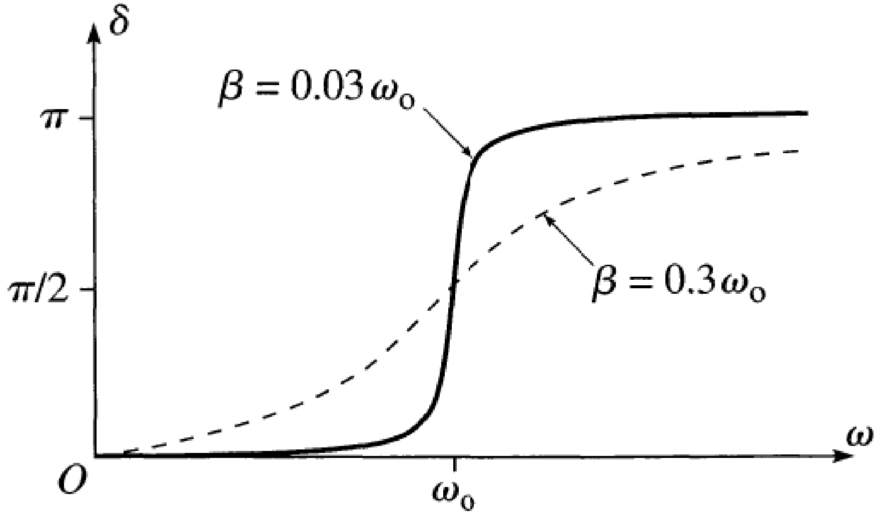


Figure 5.16 The amplitude squared, A^2, of a driven oscillator, shown as a function of the natural frequency ω_o, with the driving frequency ω fixed. The response is dramatically largest when ω_o and ω are close.



$$\delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}.$$

Physics I

Lecture 12

Reformulation of Newtonian Dynamics

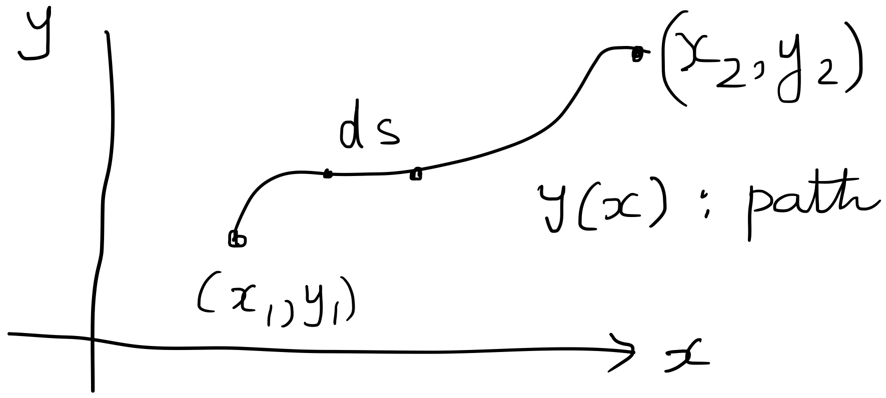
work of Lagrange & Hamilton

several advantages

Some Techniques in Calculus of Variations

Examples

- shortest path between two points in a plane



Task: to find $y = y(x)$ such that it has the shortest length between (x_1, y_1) and (x_2, y_2).

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dx\sqrt{1 + y'^2}$$

$$y' = \frac{dy}{dx}.$$

$$\boxed{L = \int_{x_1}^{x_2} dx \sqrt{1+y'^2}}$$

$\rightarrow$ find $y(x)$ such that L is minimum

Contrast with elementary calculus, where the unknown is the value of x at a pt. where $f(x)$ is minimum $\qquad \frac{df}{dx} = 0$

Ex 2. Fermat's principle in optics.

Path taken by light between two fixed pts

$\longrightarrow$ shortest time

$$\text{time of travel} = \int_1^2 dt = \int_1^2 \frac{ds}{v}$$

If we have single medium $= \frac{1}{c}\int_1^2 n\, ds = \frac{n}{c}\int_1^2 ds$

$\Longrightarrow$ same as minimum path

In general, $n = n(x,y)$.

$$\int_1^2 dt = \frac{1}{c}\int_1^2 n(x,y)\, ds = \frac{1}{c}\int_1^2 dx\, n(x,y)\sqrt{1+y'^2}$$

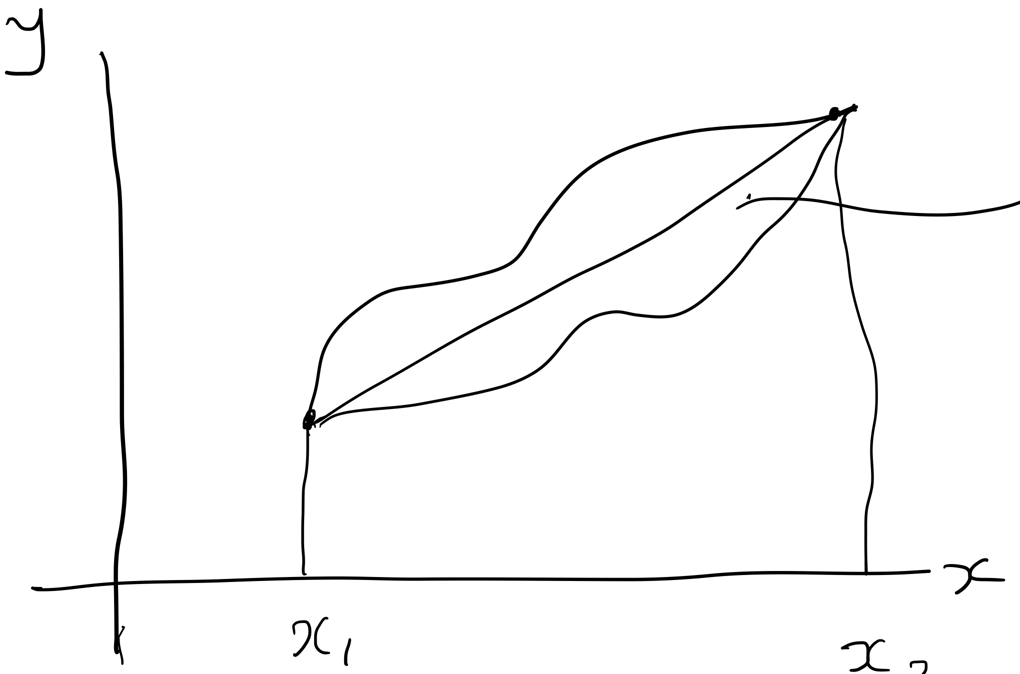
$\longrightarrow$ harder problem

Consider

$$J = \int_{x_1}^{x_2} f\left[y(x), y'(x) ; x\right] dx$$

independent variable

$$\boxed{\text{Basic problem: To determine } y(x) \text{ such that } J \text{ is an extremum.}}$$



extremum path $y(x)$.

parametric representation

$y = y(\alpha, x)$ such that

$y = y(0, x) = y(x)$

$$y(\alpha, x) = y(0, x) + \alpha \eta(x) \quad \text{—— } 1$$

continuous first derivative and vanishes at end pts

$\eta(x_1) = \eta(x_2) = 0$

Notice J is now a fn. of α

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\}\, dx \quad \text{— (2)}$$

Condition that integral have a stationary value

$$\boxed{\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0} \quad \text{— (3)}$$

$\longrightarrow$ necessary condition.

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(y, y'; x)\, dx$$

Because limits are fixed differentiation affects only the integrand

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx . \quad \text{---} (3)$$

Recall

$$y = y(0, x) + \alpha \eta(x)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) , \quad \frac{\partial y'}{\partial \alpha} = \eta'(x) \quad \text{---} (4)$$

Plug 4 into 3

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) dx .$$

Integrate 2nd term by parts

$$\int u\,dv = uv - \int v\,du .$$

$$\int \frac{\partial f}{\partial y'} \frac{d\eta}{dx} dx = \underbrace{\frac{\partial f}{\partial y'} \eta \Big|_{x_1}^{x_2}}_{0} - \int \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x)\, dy$$

η vanishes at limits

$$\int \frac{\partial f}{\partial y'} \frac{d\eta}{dx}\, dx = -\int \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x)\, dx \quad \text{--- (5)}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x)\right] dx$$

Using (5)

$$= \int_{x_1}^{x_2} \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}_{=0} \eta(x)\, dx \quad \text{--- (6)}$$

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

integrand vanishes since $\eta(x)$ is arbitrary.

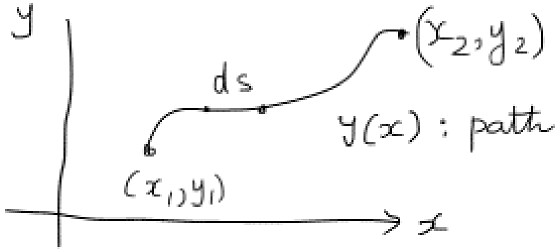
$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \Rightarrow \text{Euler Lagrange eqn.}$$

Some Techniques in Calculus of Variations

Examples

- shortest path between two points in a plane



Task: to find $y = y(x)$ such that it has the shortest length between (x_1, y_1) and (x_2, y_2).

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dx\sqrt{1 + y'^2}$$

$$y' = \frac{dy}{dx} \longrightarrow L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \quad \text{— 1}$$

Compare 1 & 2

$$J = \int_{x_1}^{x_2} f(y, y'; x)\, dx \quad \text{— 2} \qquad f = \sqrt{1 + y'^2}$$

E – L eqn:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad f = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = 0 \,, \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = \text{const} = c$$

$$y'^2(1-c^2) = c^2$$

$$y' = \pm \frac{c}{\sqrt{1-c^2}} = a$$

$$\boxed{y = ax + b}$$

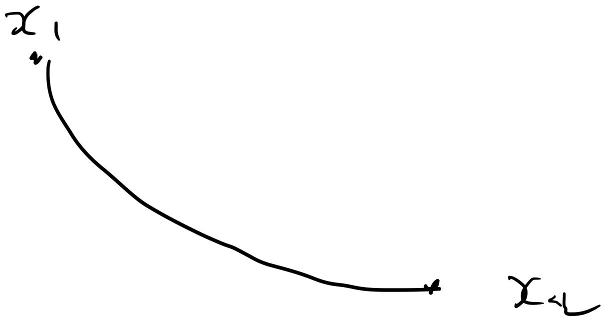
$y = y_1$
$x = x_1$
$y = y_2$
$x = x_2$

Johann Bernoulli posed the problem of the brachistochrone to the readers of *Acta Eruditorum* in June, 1696.[5][6] He said:

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise

Bernoulli wrote the problem statement as:

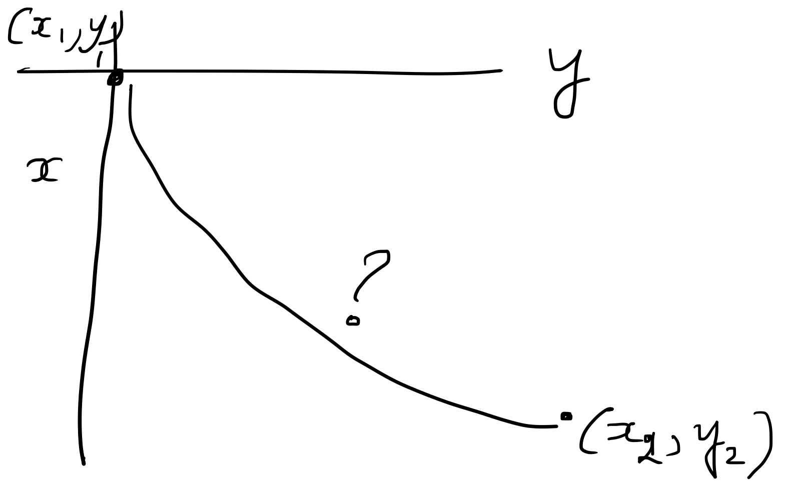
> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.



Physics I

Lecture 13

Brachistochrone problem



Path of minimum time between $(x_1, y_1) \longrightarrow (x_2, y_2)$ under gravity? released from rest.

$$E = \frac{1}{2}mv^2 - mgx = 0$$

$$\frac{1}{2}mv^2 = mgx$$

$$v = \sqrt{2gx}$$

$$t = \int_{(x_1, y_1)}^{(x_2, y_2)} \frac{ds}{v}$$

$$= \int_{x_1}^{x_2} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \int_{x_1}^{x_2} dx \frac{\sqrt{1 + y'^2}}{\sqrt{2gx}}.$$

$\sqrt{2g}$ does ~~do~~ not affect final eqn. $\longrightarrow$ $f = \left(\frac{1 + y'^2}{x} \right)^{1/2}$

$$E\text{-}L = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \quad \text{Not } \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

or $\left(\frac{\partial f}{\partial y'}\right) = \text{const} = (2a)^{-1/2}$.

$f = \left(\frac{1 + y'^2}{x}\right)^{1/2}$

$$\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \quad \text{—} (1)$$

from (1) $\quad y = \int \frac{x\,dx}{(2ax - x^2)^{1/2}} \quad \text{—} (2)$

$$\left.\begin{aligned} x &= a(1 - \cos\theta) \\ dx &= a \sin\theta \, d\theta \end{aligned}\right\} \quad \text{—} (3)$$

$$x = a(1-\cos\theta) \quad , \quad dx = a\sin\theta \, d\theta \Big\} - (3)$$

substituting (3) in (2)

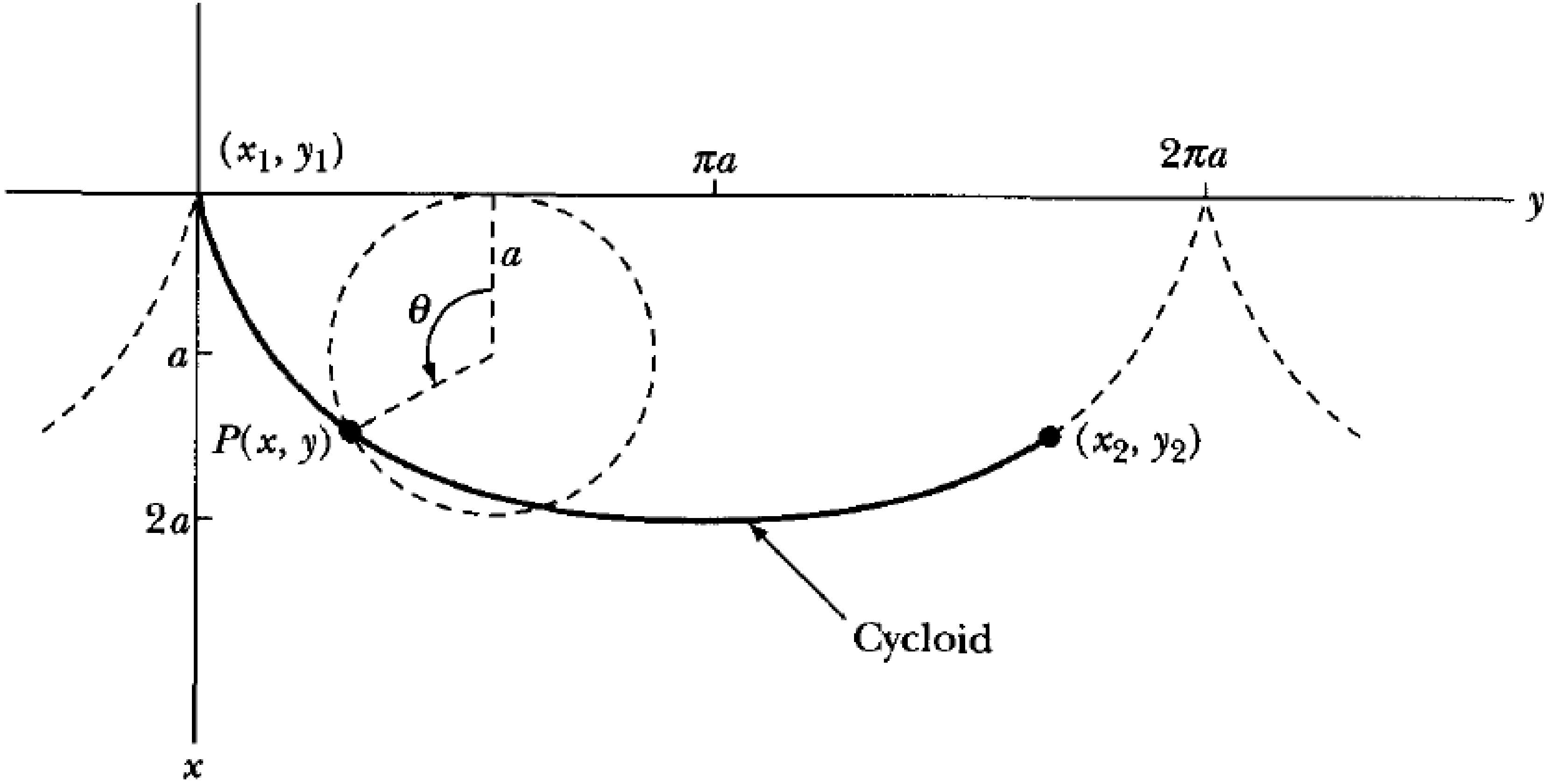
$$y = a\int(1-\cos\theta)\,d\theta = a(\theta - \sin\theta) + \text{const}.$$

$$\boxed{\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}}$$

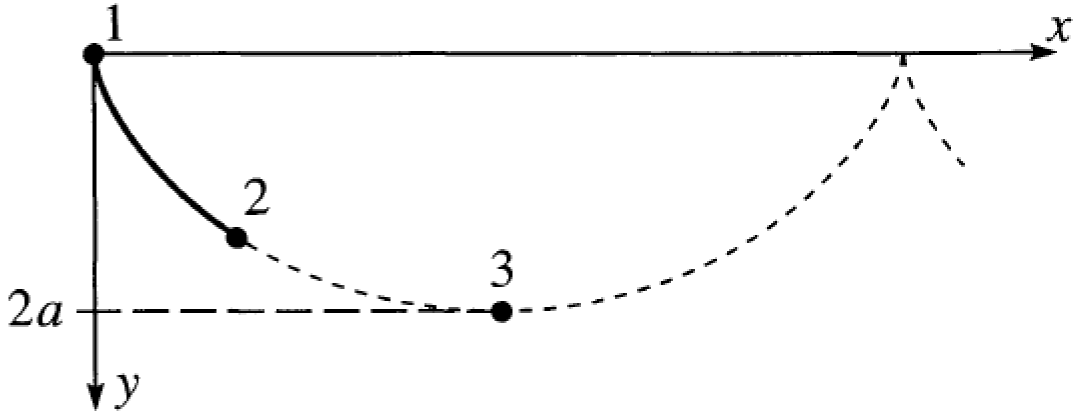
const = 0

(0, 0) starting pt.

a has to be adjusted to allow curve to pass through (x_2, y_2).



(x_1, y_1)
πa
$2\pi a$
y
a
θ
a
$P(x, y)$
(x_2, y_2)
$2a$
Cycloid
x



Time period independent of amplitude
isosynchronous.

Physics I

Lecture 14

Recap.

$$J = \int_{x_1}^{x_2} f\left[y(x), y'(x); x\right] dx$$

the path that extremezis it, $y(x)$

is determined by the E-L eqn.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

Generalization to multivariable case

$$f\left[y_1(x), y_1'(x), y_2(x), y_2'(x) \cdots ; x\right]$$

$$f\left[y_i(x), y_i'(x) ; x\right] \qquad i = 1 \cdots n$$

$$y_i(x) = y_i(0,x) + \alpha \eta_i(x)$$

following 1 variable derivation

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right) \eta_i(x)\, dx$$

η_i's are independent $\quad \partial J / \partial \alpha = 0$

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \longrightarrow \text{E-L eqns}$$

Hamilton's Principle & Lagrangian Dynamics

- experience shows that in inertial frames particle motion is correctly described by Newton's Laws
  $\vec{F} = \dot{\vec{p}}$

- Practical difficulties in applying Newton's Laws.
  e.g. non Cartesian coordinates $\rightsquigarrow$ motion on a sphere
  projection of vector eqns on the sphere is complicated

- Constraints, example bead sliding on wire
  forces of constraint are complicated and occasionally cannot get explicit expressions. $\vec{F}$ includes all forces.

Alternative formulation of Newtonian dynamics

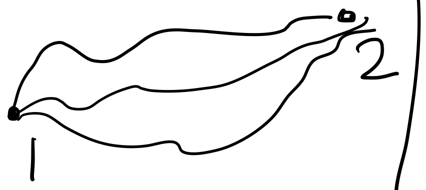
Lagrange Eqns.

$$\boxed{\text{Hamilton's Principle}} \equiv \text{Newton's Laws} .$$

elegant, applicable to a wide variety of physical phenomena including field

We will stick to conservative systems .

Hamilton's Principle



The actual path which a particle follows between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} L \, dt$$

is stationary when taken along the actual path

where $L = T - U$; L: Lagrangian

kinetic energy

potential energy.

Summary

In mechanics (conservative forces only):

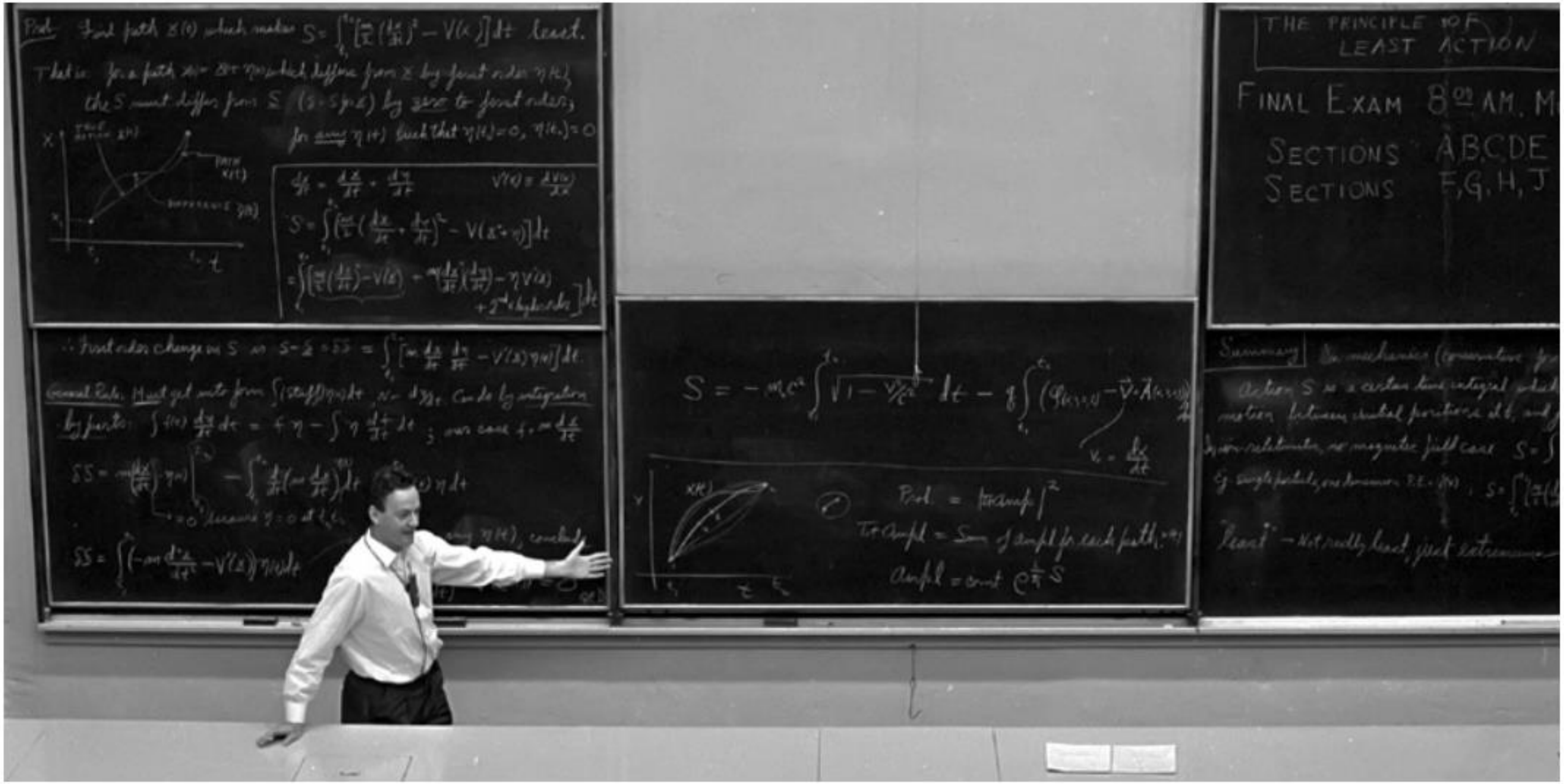
Action S is a certain time integral which is "least*" for the true motion between initial positions at t_1 and final ones at t_2.

NOT +

In non-relativistic, no magnetic field case $S = \int (\text{Kinetic Energy} - \text{Potential Energy})\,dt$

Eg. single particle, one dimension P.E. = $V(x)$; $S = \int_{t_1}^{t_2} \left[\frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 - V(x(t))\right] dt.$

"least*" → Not really least, just extremum ← means first order change = 0.


THE PRINCIPLE OF LEAST ACTION
FINAL EXAM
SECTIONS
SECTIONS
Summary

$$S = \int_{t_1}^{t_2} L(x_i, \dot{x}_i; t)\, dt \quad \text{—— (1)}$$

Recall $$J = \int_{x_1}^{x_2} f[y_i, y_i'; x]\, dx \quad \text{—— (2)}$$

make the correspondence

$$\left.\begin{aligned} x &\longrightarrow t \\ y_i(x) &\longrightarrow x_i(t) \\ y_i'(x) &\longrightarrow \dot{x}_i(t) \end{aligned}\right\}$$

Euler-Lagrange eqn.

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0}$$

Examples

① Free particle in 3D.

$$L = T - U \quad , \quad U = 0 .$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) .$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0$$

$$\frac{d}{dt}\left(m\dot{x}\right) = 0 = \frac{d}{dt}\left(m\dot{y}\right) = \frac{d}{dt}\left(m\dot{z}\right)$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0 .$$

Ex 2 1-d Harmonic Oscillator

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 .$$

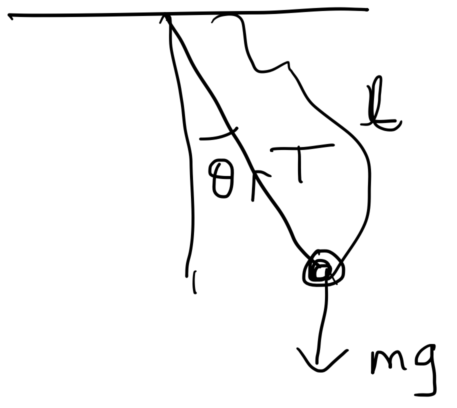
$$\frac{\partial L}{\partial x} = -kx \quad ; \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 .$$

$$m\ddot{x} + kx = 0$$

Ex 3

Plane pendulum



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta).$$

E-L eqn.

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

identical to Newtonian result.

Generalized Coordinates

Degrees of freedom : In general for n (free) particles is 3n.

m constraints

degrees of freedom $\boxed{s = 3n - m}$

Need to choose s coordinates to describe motion
need not be Cartesian coordinates (can choose curvilinear coordinates, spherical, cylindrical)
need not even have dimensions of length.

generalized coordinates $\{q_i\} \rightarrow$ not unique generalized coordinates $\{q_j\}$ + generalized velocities $\{\dot{q}_j\}$

Coordinate transformations

$$\boxed{\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, \cdots q_s, t) \\ &= x_{\alpha,i}(q_j, t) \\ \\ \dot{x}_{\alpha,i} &= \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \end{aligned}}$$

where $\alpha = 1, \cdots n$

$i = 1, 2, 3$

$j = 1, \cdots s$.

$m = 3n - s$.

Inverse transform

$$\left.\begin{aligned} q_j &= q_j(x_{\alpha,i}, t) \\ \dot{q}_j &= \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{aligned}\right\}$$

+ constraint eq$^{\underline{n}}$

$f_k(x_{\alpha,i}, t) = 0$

$k = 1, \cdots m$

Physics I

Lecture 15

Hamilton's Principle

$$S = \int_{t_1}^{t_2} L \, dt$$

$$\boxed{\delta S = 0}$$

$$\delta S \equiv \frac{\partial S}{\partial \alpha} d\alpha$$

$$L = T - U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \implies \text{E-L eqn.}$$

generalized coordinates $\{q_j\}$ + generalized velocities $\{\dot{q}_j\}$

Coordinate transformations

$$\boxed{\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, \cdots q_s, t) \\ &= x_{\alpha,i}(q_j, t) \\ \dot{x}_{\alpha,i} &= \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \end{aligned}}$$

where $\alpha = 1, \cdots n$

$i = 1, 2, 3$

$j = 1, \cdots s$.

$m = 3n - s$.

Inverse transform

$$\left.\begin{aligned} q_j &= q_j(x_{\alpha,i}, t) \\ \dot{q}_j &= \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{aligned}\right\}$$

+ constraint eq$^{\underline{n}}$

$f_k(x_{\alpha,i}, t) = 0$

$k = 1, \cdots m$

Lagrangian is a scalar $\Rightarrow$ coordinate invariant

$$L = T(\dot{x}_{\alpha,i}) - U(x_{\alpha,i}) = T(q_j, \dot{q}_j, t) - U(q_j, t)$$

$$L = L(q_j, \dot{q}_j, t) \Rightarrow \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0$$

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0} \qquad j = 1, 2, \cdots s.$$

Is the Lagrangian unique? What degree of arbitrariness does it have?

for example U is not unique $U \to U + \text{constant}$

$\Longrightarrow$ does not change eqns of motion.

It turns out that L is arbitrary upto

$$L' \longrightarrow L + \frac{d}{dt} f(q_i, t)$$

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt + \int_{t_1}^{t_2} \frac{df}{dt}\, dt$$

$$= S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1).$$

$$S' = S + \underbrace{f(q^{(2)}, t_2) - f(q^{(1)}, t_1)}_{} .$$

→ does not vary on variation, end pts fixed.

↓ notice

$$\delta S' = \delta S$$

↓ Leads to identical E-L eqns of motion.

Equivalence of Lagrange & Newton's eqns

choose generalized coordinates as Cartesian coordinates

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \qquad i = 1, 2, 3$$

$$\frac{\partial (T-U)}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{x}_j}\right) = 0 .$$

For conservative system in rectangular coordinates

$$T = T(\dot{x}_i) \quad \text{and} \quad U = U(x_i) .$$

$$\therefore \quad \frac{\partial T}{\partial x_i} = 0 \quad , \quad \frac{\partial U}{\partial \dot{x}_i} = 0$$

$$\frac{\partial}{\partial x_j}(T-U) - \frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{x}_j}\right] = 0 \quad \Big\} \quad T = \sum_{j=1}^{3} \frac{1}{2} m \dot{x}_i^2$$

$$\frac{\partial T}{\partial \dot{x}_j} = m \dot{x}_i$$

$$-\frac{\partial U}{\partial x_j} - \frac{d}{dt}\left[m \dot{x}_j\right] = 0$$

But we know $-\frac{\partial U}{\partial x_j} = F_j$

$$\boxed{m \ddot{x}_j = F_j} \implies \text{recover Newton's Law.}$$

Application of Lagrangian formulation

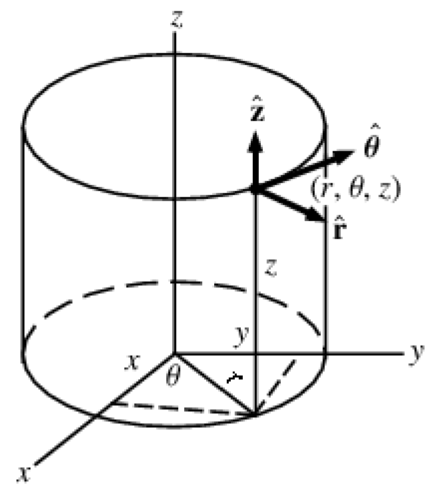
1. Free particle . $V = 0$

$$L = \frac{1}{2} m v^2$$

choose generalized coordinates (x, y, z) Rectangular

Cartesian Coordinates

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

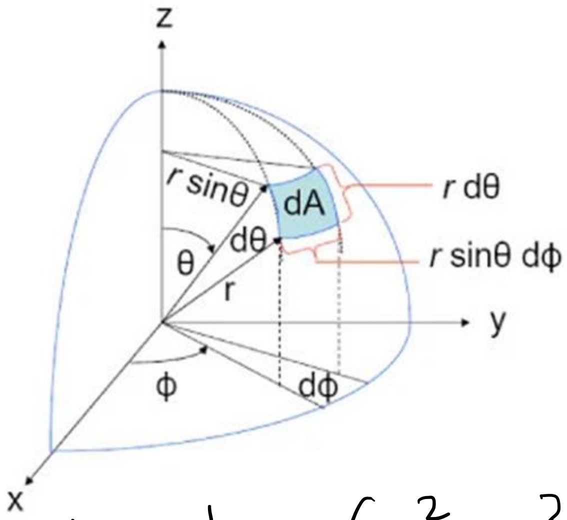


Cylindrical coordinates.

$$\vec{r} = (r, \theta, z)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$



Spherical Polar coordinates

$$\vec{r} = (r, \theta, \phi)$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

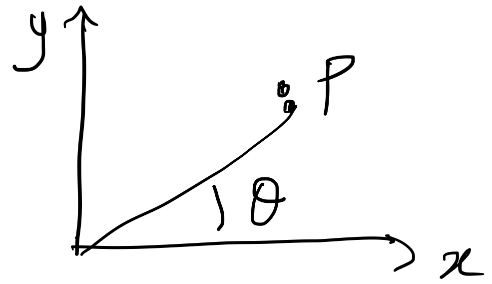
2D Polar coordinates

x, y

$m\ddot{x} = 0$

$m\ddot{y} = 0$.

free particle



$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

E-L eqns. (r, θ)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt}(m\dot{r}) - m r \dot{\theta}^2 = 0$$

$$\boxed{m\ddot{r} = m r \dot{\theta}^2} \quad \text{— (1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \frac{d}{dt}(m r^2 \dot{\theta}) = 0 \quad \text{— (2)}$$

L independent of θ $\longrightarrow$ $\boxed{m r^2 \dot{\theta} = \text{const}}$

Ang mom conserved.

Conservation Laws

cyclic coordinate.

Lagrangian is independent of this coordinate

q_k

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

0 ; q_k cyclic.

In our previous ex θ was cyclic

Ang mom conserved

$$\boxed{\frac{\partial L}{\partial \dot{q}_k} = \text{conserved}.}$$

Generalized momentum is conserved.

$$\frac{\partial L}{\partial \dot{q}_k}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

2D Cartesian coordinates (x, y)

cyclic coordinates?

$\downarrow$ (x, y)

both cyclic.

E-L eqn.

$\downarrow$

Generalized mom x $\qquad \dfrac{\partial L}{\partial \dot{x}} = \underbrace{m\dot{x}}_{p_x} \longrightarrow$ conserved

Gen. mom for y $\qquad \dfrac{\partial L}{\partial \dot{y}} = \underbrace{m\dot{y}}_{p_y} \longrightarrow$ conserved.

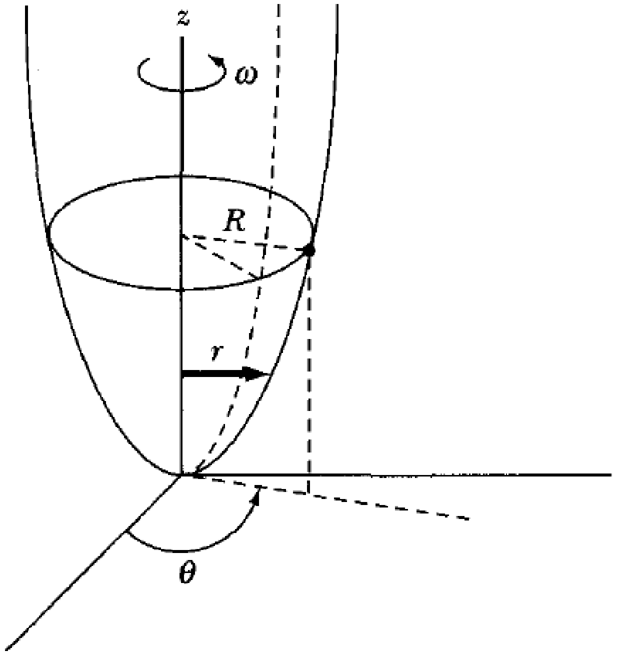
Physics I

Lecture 16

Recap

- E-L eqns $\equiv$ Newton's Laws

- If q_K is a cyclic coordinate, the corresponding $\underbrace{\text{generalized momentum}}_{\frac{\partial L}{\partial \dot{q}_K}}$ is conserved.



A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$. The bead rotates in a circle of radius R when the wire is rotating about its vertical with angular vel. ω. Find the value of c

generalized coordinates
(r, θ, z)

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right]$$

$U = 0, \; z = 0$

$U = mgz$.

$z = cr^2$.

$\dot{z} = 2c\dot{r}r$

$\theta = \omega t \quad \dot{\theta} = \omega$.

$$L = T - U$$

$$= \frac{m}{2}\left[\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2\right] - mgz.$$

Plug in the constraints

$$= \frac{m}{2}\left[\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right] - mgcr^2.$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}\left(2\dot{r} + 8c^2r^2\dot{r}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{m}{2}\left(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r}\right) \quad \text{— } 1$$

$$\frac{\partial L}{\partial r} = m\left(4c^2r\dot{r}^2 + r\omega^2 - 2gcr\right) \quad \text{— } 2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

Plugging in from 1 & 2

$$\ddot{r}\left(1 + 4c^2r^2\right) + \dot{r}^2\left(4\,{}^2r\right) + r\left(2gc - \omega^2\right) = 0 .$$

But $r = R$, kills $\ddot{r}$, $\dot{r}$ terms.

$$R\left(2gc - \omega^2\right) = 0$$

$$\boxed{c = \frac{\omega^2}{2g}}$$

New look at Conservation Laws

A Theorem concerning K.E (n particles in 3D).

K.E in fixed rectangular coordinates.

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2 \quad \text{—— } (3)$$

Now let us transform to generalized coordinates and velocities.

m constraints $m = 3n - s$.

$s = 3n - m$

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t) \qquad j = 1, \dots s. \quad \text{—— } (4)$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t}. \quad \text{—— } (5)$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j}\,\dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t} \quad — \quad (5)$$

Plug (5) into (3)

$$T = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{i=1}^{3} m_\alpha \,\dot{x}_{\alpha,i}^{2} \quad — \quad (3)$$

$$x_{\alpha,i}^{2} = \sum_{j,k} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\,\dot{q}_j\,\dot{q}_k + 2\sum_{j} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\,\dot{q}_j$$

$$+ \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \quad — \quad (6)$$

$$T = \sum_{\alpha}\sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\,\dot{q}_j\,\dot{q}_k + 2\sum_{\alpha,i,j,k} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\,\dot{q}_j$$

$$+ \sum_{\alpha}\sum_{i} \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \quad — \quad (7)$$

Can rewrite (7) as.

$$T = \sum_{j,k} a_{jk}\,\dot{q}_j\,\dot{q}_k + \sum_j b_j\,\dot{q}_j + c \quad — (8)$$

Special case, when the transformation does not explicitly depend on time

$$\frac{\partial x_{\alpha,i}}{\partial t} = 0 \quad — (9) \implies b_j = 0\,,\ c = 0$$

Under these conditions, kinetic energy is a homogeneous quadratic fn. of the generalized velocities.

Note that

$a_{jk} = a_{kj}$

$$\boxed{T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k} \quad (10)$$

Differentiate T w.r.t $\dot{q}_\ell$, [Are you familiar with $\delta_{ij} = ?$]

$$\boxed{\begin{aligned} \delta_{ij} &= 0 \quad i \neq j \\ &= 1 \quad i = j \end{aligned}}$$

Note. $\boxed{\dfrac{\partial \dot{q}_j}{\partial \dot{q}_\ell} = \delta_{j\ell}}$

$$\frac{\partial T}{\partial \dot{q}_\ell} = \sum_{j,k} a_{jk} \delta_{j\ell} \dot{q}_k + \sum_{j,k} a_{jk} \dot{q}_j \delta_{k\ell}$$

$$= \sum_k a_{\ell k} \dot{q}_k + \sum_j a_{j\ell} \dot{q}_j$$

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_k a_{lk} \dot{q}_k + \sum_j a_{jl} \dot{q}_j$$

k, j are dummy indices

$$= \sum_k a_{lk} \dot{q}_k + \sum_k a_{lk} \dot{q}_k \qquad [a_{jl} = a_{lj}]$$

$$= 2 \sum_k a_{lk} \dot{q}_k$$

$$\boxed{\sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2 \sum_{l,k} a_{lk} \dot{q}_k \dot{q}_l = 2T} \tag{11}$$

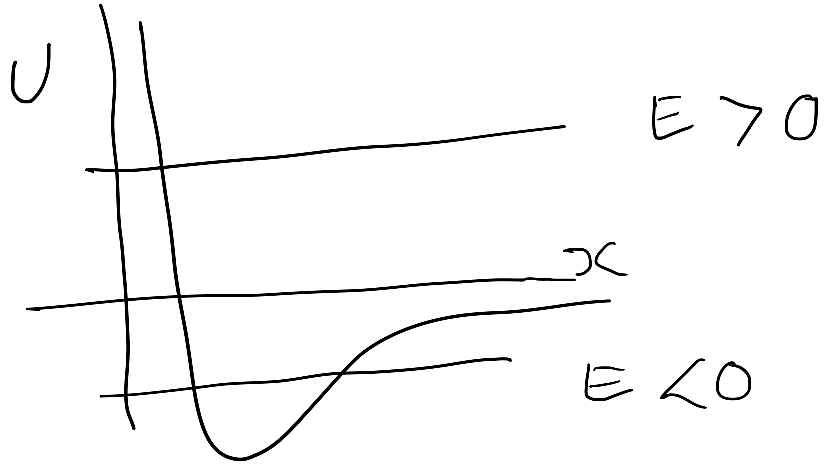
special case of Eulers theorem, $f(y_k)$ is a homogeneous fn. of y_k of degree n

$$\sum_k y_k \frac{\partial f}{\partial y_k} = nf$$

Physics 1

Lecture 17

- Always plot the potential



- In problems like

$$U(x) = \frac{\alpha}{x} - \frac{\beta}{x^2}$$

always convenient to take ∞ as reference pt.

- In problem 5 $F_x = ayz + bx + c$, $F_y = axz + bz$

$F_z = \ldots$

$$\vec{\nabla} \times \vec{F} = 0$$

$$U = -\int \vec{F} \cdot d\vec{r} = \int F_x \, dx + \int F_y \, dy + \int F_z \, dz$$

Remember, line integral.

$$W_a = \int_a \mathbf{F} \cdot d\mathbf{r} = \int_O^Q \mathbf{F} \cdot d\mathbf{r} + \int_Q^P \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_x(x, 0)\, dx + \int_0^1 F_y(1, y)\, dy$$

$$= 0 + 2\int_0^1 dy = 2.$$

$$\mathbf{F} = (y, 2x)$$

example

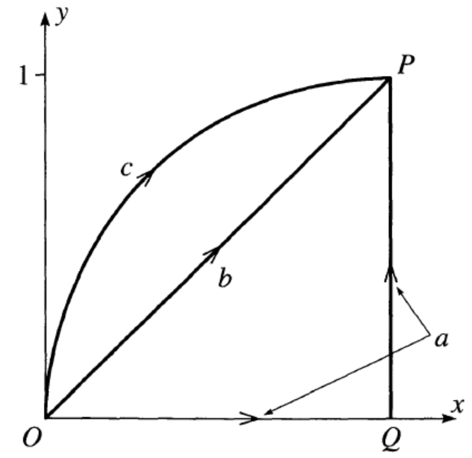


Figure 4.2 Three different paths, a, b, and c, from the origin to the point $P = (1, 1)$.

On the path b, $x = y$, so that $dx = dy$, and

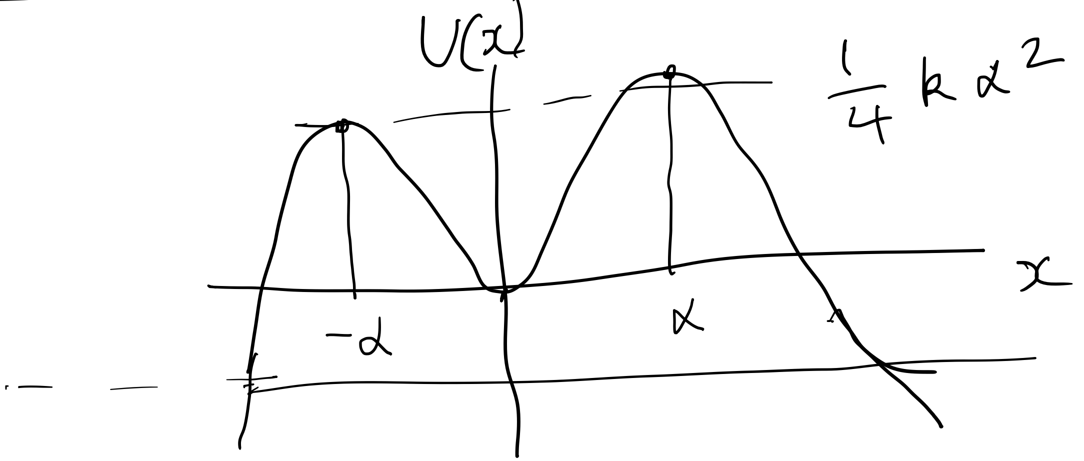
$$W_b = \int_b \mathbf{F} \cdot d\mathbf{r} = \int_b (F_x\, dx + F_y\, dy) = \int_0^1 (x + 2x)dx = 1.5.$$

- $\vec{F} = \frac{a}{r}\hat{r}$ (a, b, c) are constants.

$F_x = \frac{a^2}{r}$, $F_y = \frac{ab}{r}$, $F_z = \frac{ac}{r}$ X .

Problem 2

$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}$



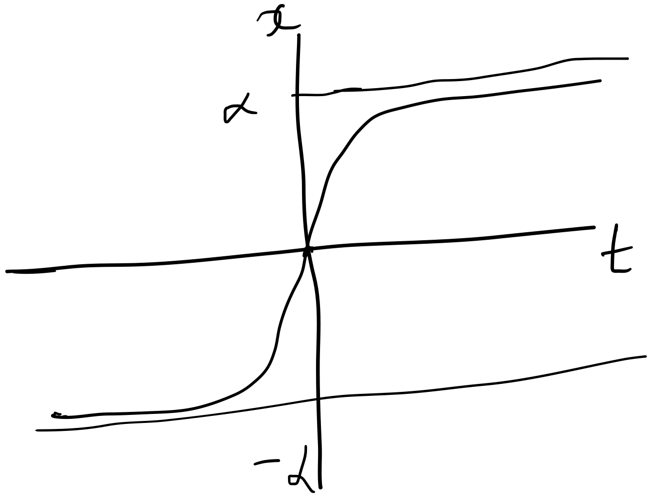
$\frac{1}{4}k\alpha^2 = E$; Are $\alpha, -\alpha$ turning pts ?

$E = U$ at these pts.

not turning pts.

pts of unstable equi.

$$U(x) = \frac{1}{2} k x^2 - \frac{1}{4} \frac{k x^4}{\alpha^2}$$



$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 - \frac{1}{4} \frac{k x^4}{\alpha^2} .$$

$$E = \frac{1}{4} k \alpha^2 .$$

$$\dot{x}^2 = \frac{1}{4} \frac{k x^4}{\alpha^2} - \frac{1}{2} k x^2 + \frac{1}{4} k \alpha^2 .$$

$$\sqrt{\frac{2m}{k}} \int \frac{dx}{(\alpha^2 - x^2)} = \int dt$$ can be exactly

$$\hookrightarrow \quad x = \alpha \tanh\left(\sqrt{\frac{k}{2m}} \alpha t \right)$$

Recap.

$$\boxed{\sum_{\ell} \dot{q}_{\ell} \frac{\partial T}{\partial \dot{q}_{\ell}} = 2T}$$

provided $x_{\alpha,i} = x_{\alpha,i}(q_j, \cancel{t})$

Conservation Laws & Symmetries

time is homogeneous within an inertial coordinate system. $\longrightarrow$ symmetry

$\therefore$ **Lagrangian of a closed system cannot depend explicitly on time**

$$\frac{\partial L}{\partial t} = 0 \qquad \text{—— } (1) \qquad\qquad L(q_j, \dot{q}_j ; t).$$

$$\frac{dL}{dt} = \sum_{j=1}^{s} \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_{j=1}^{s} \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \cancelto{0}{\frac{\partial L}{\partial t}} \qquad \text{—— } (2)$$

Use Euler-Lagrange eqn. in (2)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{—— } (3)$$

$$\frac{dL}{dt} = \sum_{j} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j + \sum_{j} \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \qquad \text{—— } (4)$$

Can rewrite (4) as

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) \quad — (5)$$

or $$\frac{d}{dt}\Big\{\underbrace{L - \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j}_{-H}\Big\} = 0 .$$

Hamiltonian

$$\Rightarrow \quad \frac{dH}{dt} = 0 \qquad \boxed{H = \text{const}} \quad — (6)$$

If the potential energy $U(x)$ does not explicitly on the velocities $\dot{x}_{\alpha,i}$ or t,

$$U = U(x_{\alpha,i})$$

the coordinate transformations will be of the form

$$x_{\alpha,i} = x_{\alpha,i}(q_j) \quad \text{or} \quad q_j = q_j(x_{\alpha,i}).$$

$$U = U(q_j), \quad \frac{\partial U}{\partial \dot{q}_j} = 0$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \quad \text{——} \quad (7)$$

So now

$$-H = L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}$$

$$= (T-U) - \underbrace{\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}}_{2T}$$

$$= T - U - 2T = -(T+U)$$

$$= -E$$

$\boxed{H = E}$ $\longrightarrow$ energy conserved

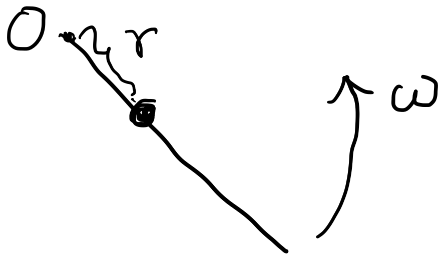
$H = E$ only if certain conditions are met

1. The eqns of transfn of coordinate must be independent.

2. The potential must be independent of velocity $\longrightarrow \dfrac{\partial U}{\partial \dot{q}_j} = 0$

Bead on Stick:

A stick is pivoted at the origin and is arranged to swing around in a horizontal plane with constant angular speed ω. A bead of mass m slides frictionlessly along the stick. Let r be the radial position of the bead. Find the Hamiltonian. Explain why this is **not** the energy of the bead.



No potential energy, only K.E

$$\dot{\theta} = \omega$$

$$L = T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \omega^2$$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= \frac{\partial L}{\partial \dot{r}} \dot{r} - \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \omega^2$$

$$H = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2 \neq E \qquad \frac{\partial L}{\partial t} = 0$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 .$$

Physics I

Lecture 18

Homogeneity of time $\longrightarrow$ $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \quad H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad \Rightarrow \text{conserved}$$

when $x_{i,\alpha} = x_{i,\alpha}(q_j)$ not time dependent

$$H = E$$

Closed inertial system

Homogeneity of space $\longrightarrow$ all points of space are equivalent

$\hookrightarrow$ The Lagrangian of the system is invariant under a translation of the entire system in space.

$$\vec{r}_\alpha \Rightarrow \vec{r}_\alpha + \underline{\delta \vec{r}_\alpha} = \vec{r}_\alpha + \vec{\epsilon} \qquad \text{—— (1)}$$

clearly $\quad \delta \dot{\vec{r}}_\alpha = 0$

$$\delta L = \sum_\alpha \sum_i \frac{\partial L}{\partial x_{\alpha,i}} \cdot \delta x_{\alpha_i} + \sum_\alpha \sum_i \frac{\partial L}{\partial \dot{x}_{\alpha,i}} \underbrace{\delta \dot{x}_{\alpha,i}}_{=0}$$

$$= \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta \vec{r}_\alpha \qquad \text{—— (2)}$$

$$\delta L = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta \vec{r}_\alpha .$$

$$= \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \vec{\epsilon}$$

$$= \vec{\epsilon} \cdot \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} .$$

If L is invariant under the transfn. $\delta L = 0$

$$\delta L = 0 \Rightarrow \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

$\vec{\epsilon}$ arbitrary

$L = T - U$

$$\equiv \sum_\alpha \vec{F}_\alpha = 0$$

$$\sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

E-L equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) - \frac{\partial L}{\partial \vec{r}_\alpha} = 0$

$$\rightarrow \sum_\alpha \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) = 0 \quad , \quad \frac{d}{dt} \sum_\alpha \Big(\underbrace{\frac{\partial L}{\partial \vec{v}_\alpha}}_{\vec{p}_\alpha}\Big) = 0 .$$

$$\frac{d}{dt}\left(\sum_\alpha \vec{p}_\alpha\right) = 0$$

$$\Longrightarrow \boxed{P = \sum_\alpha \vec{p}_\alpha} \text{ conserved}$$

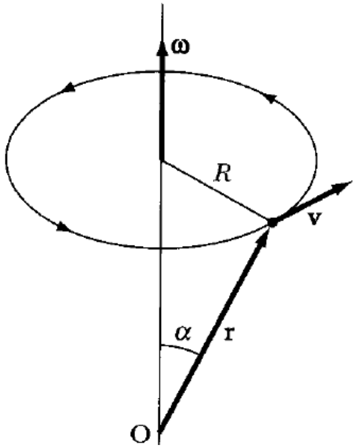
Generalized coordinates

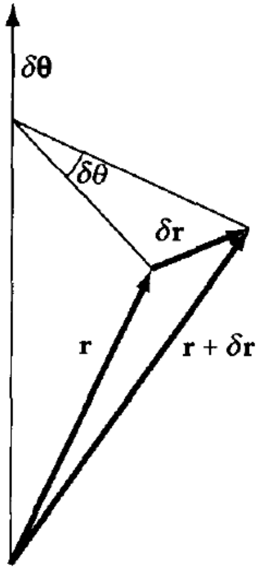
$$\sum_{\alpha, i} \frac{\partial L}{\partial \dot{q}_{i,\alpha}} = \text{conserved}$$

Next symmetry

Isotropy of space.

$\hookrightarrow$ Lagrangian invariant under rotations.





A particle moving arbitrarily in space, can always be considered, *at a given instant,* to be moving in a plane circular path about a certain axis. That is, the path that a particle describes during an infinitesimal time interval δt is represented by an infinitesimal arc of a circle. The line passing through the centre and perpendicular to the instantaneous direction of motion is called the instantaneous axis of rotation.

$$\boxed{\delta\vec{r} = \overrightarrow{\delta\theta} \times \vec{r}}$$

$$\omega = \frac{d\theta}{dt} \qquad \dot{\vec{r}} = \vec{v}$$

$$\vec{v} = \vec{\omega} \times \vec{r} \quad ; \quad v = r\sin\alpha\,\omega$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$$

$$\delta\dot{\vec{r}} = \delta\vec{\theta} \times \dot{\vec{r}}$$

E-L

$$\frac{\partial L}{\partial \vec{r}_\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) = \frac{d\vec{p}_\alpha}{dt}$$

$$\delta L = \sum_\alpha \left(\frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta\vec{r}_\alpha + \frac{\partial L}{\partial \vec{v}_\alpha} \cdot \delta\vec{v}_\alpha \right)$$

$$= \sum_\alpha \left(\dot{\vec{p}}_\alpha \cdot \delta\vec{r}_\alpha + \vec{p}_\alpha \cdot \delta\vec{v}_\alpha \right)$$

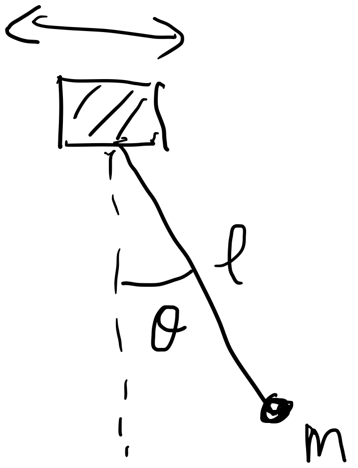
$$= \sum_\alpha \left[\dot{\vec{p}}_\alpha \cdot \delta\vec{\theta} \times \vec{r}_\alpha + \vec{p}_\alpha \cdot \delta\vec{\theta} \times \dot{\vec{r}}_\alpha \right]$$

$$= -\delta\vec{\theta} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha \times \vec{p}_\alpha \right) = 0$$

$$\Rightarrow \frac{d}{dt} \sum_\alpha \left(\vec{r}_\alpha \times \vec{p}_\alpha \right) = 0 \Longrightarrow \boxed{\vec{L} = \text{const}}$$

<u>Noether's Theorem</u> : Every continuous symmetry of a Lagrangian corresponds to a conserved quantity.

A pendulum consists of a mass m and a massless stick of length l. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos\omega t$. What is the general solution for the angle of the pendulum as a function of time ? You are allowed to make a small angle approximation.



Coordinates of mass (X, Y)

$$(X, Y) = (x + l\sin\theta, -l\cos\theta)$$

to find K.E, find V^2.

$$V^2 = \dot{X}^2 + \dot{Y}^2 = l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta$$

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + mgl\cos\theta$$

$$L = \frac{1}{2} m \left(l^2 \dot{\theta}^2 + \dot{x}^2 + 2 l \dot{x} \dot{\theta} \cos\theta \right) + mgl\cos\theta$$

E-L eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(m l^2 \dot{\theta} + m l \dot{x} \cos\theta \right) = - m l \dot{x} \dot{\theta} \sin\theta - mgl\sin\theta$$

$$\Rightarrow l \ddot{\theta} + \ddot{x} \cos\theta = - g \sin\theta \qquad x = A\cos\omega t$$

$$l \ddot{\theta} - A\omega^2 \cos\omega t \cos\theta + g\sin\theta = 0$$

small angle approx

$$\ddot{\theta} + \omega_0^2 \theta = a\omega^2 \cos\omega t \qquad \omega_0^2 = g/l$$

$$a = A/L$$

$$\ddot{\theta} + \omega_0^2 \theta = a\omega^2 \cos\omega t \longrightarrow \text{Driven oscillator}$$

$$\theta(t) = \underbrace{\frac{a\omega^2}{\omega_0^2 - \omega^2} \cos(\omega t)}_{\text{particular solution}} + \underbrace{C \cos(\omega_0 t + \phi)}_{\text{homogeneous}}$$

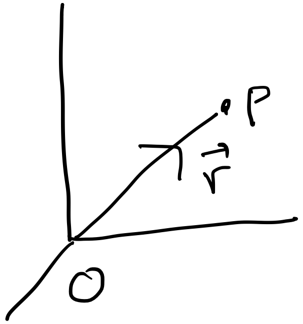
Physics I

Lecture 19

Central Force Dynamics

Motion of a two body system affected by a force along the line joining their centres.

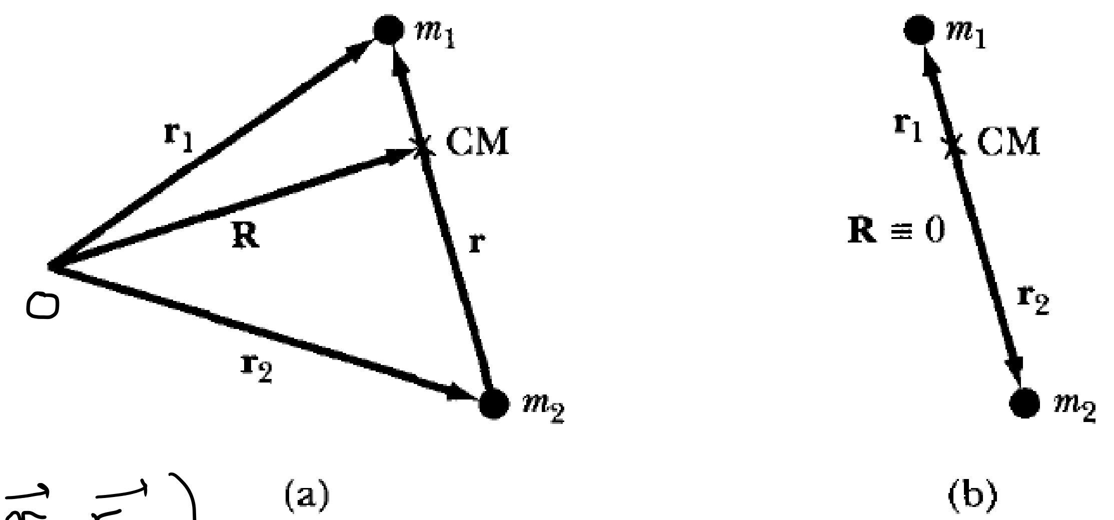
↳ motion of planets, moons, comets, ... Rutherford scattering etc.



$$\vec{F} = F(r)\,\hat{r} \longrightarrow \text{Conservative force.}$$

↳ $\vec{\nabla} \times \vec{F} = 0$

$\therefore$ $U(r)$ exists, $\vec{F} = -\vec{\nabla} U$



$(\vec{r}_1, \vec{r}_2)$

alternatively

$$(\vec{R}, \vec{r}) \longrightarrow \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \quad , \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$r = |\vec{r}_1 - \vec{r}_2| .$$

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r) .$$

Transforming coordinates to $(\vec{r}, \vec{R})$.

$$\boxed{M = m_1 + m_2} \qquad \mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r) \qquad \boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

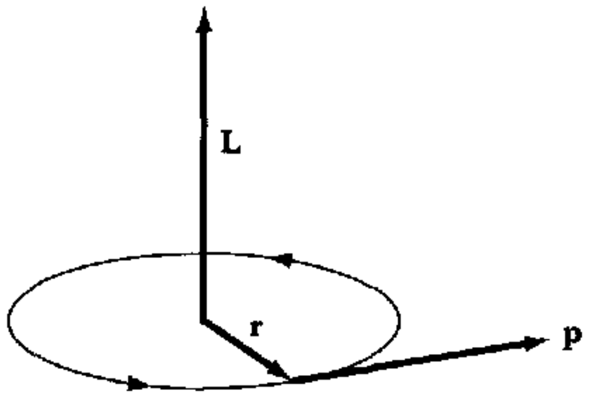
reduced mass.

$\vec{R} = 0$, $\dot{\vec{R}} = 0$ CM frame.

$$\boxed{\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)} \longrightarrow$$ effective one-body problem with mass μ.

Conserved Quantities

- Energy is conserved
- $\mathcal{L}$ is spherically symmetric, θ, ϕ both cyclic. corresponding generalized momenta are conserved



$$\vec{F} = F(r)\,\hat{r}$$

Torque $\vec{N} = \vec{r} \times \vec{F} = 0$, Angular momentum is conserved.

direction of $\vec{L}$ is const

$\vec{L} = \vec{r} \times \vec{p}$ $\quad$ $\vec{L}$ is $\perp$ to $\vec{r}, \vec{p}$, plane containing $\vec{r}, \vec{p}$

$\boxed{\text{motion is planar}}$

Can use 2d polar coordinates

$$\boxed{\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)}$$

θ is cyclic

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

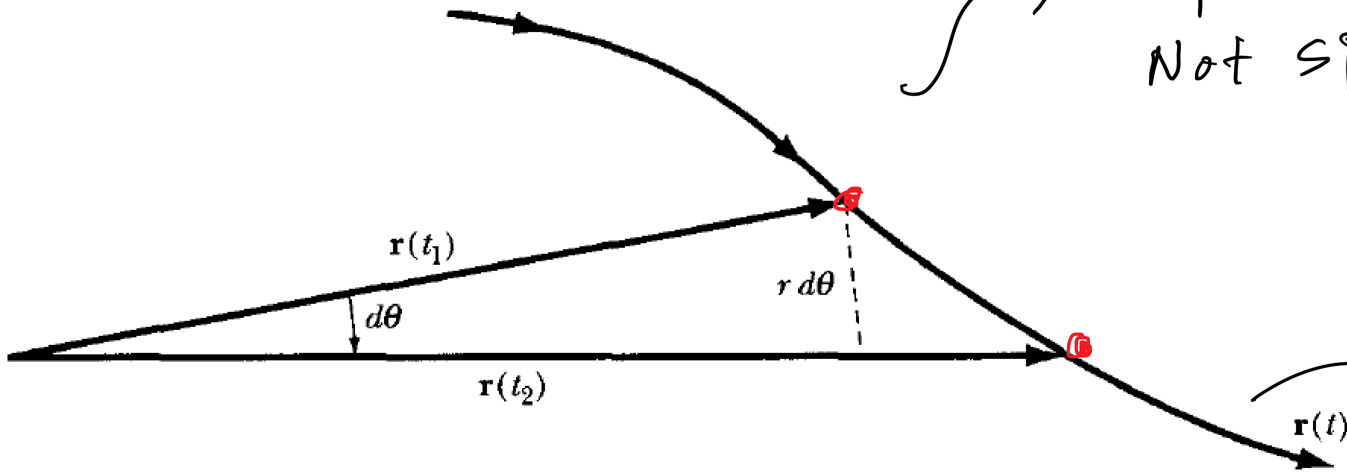
$$\boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const}}$$

Ang mom conservation

$$l = \mu r^2 \dot{\theta} = \text{const}$$

ang mom.

Kepler's 2^{nd} Law
Not specific to $\frac{1}{r^2}$ force



trajectory

Geometrical interpretation

Area swept out by radius vector in time dt

$$dA = \frac{1}{2} r \cdot r\, d\theta = \frac{1}{2} r^2 d\theta$$

$$\begin{cases} l = \mu r^2 \dot{\theta} \\ \dot{\theta} = \dfrac{l}{\mu r^2} \end{cases}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2\mu} = \text{const}$$

$$\boxed{\frac{dA}{dt} = \text{const}}$$

Energy is conserved

$$E = T + U = \text{const}$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)$$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)}$$ Using $l = mr^2\dot{\theta}$

effectively 1-d problem.

Recall that a 1-d problem is in principle solvable completely

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r).$$

Using above to solve for $\dot{r}$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}} \quad \text{---} \; * $$

$$t = \pm \int \frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}}$$

$t = t(r)$

$\hookrightarrow$ invert to get $r(t)$

Our interest is to find the trajectory $r(\theta)$.

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \qquad \left\{ \dot{\theta} = \frac{l}{\mu r^2} \right\}$$

$$= \frac{\frac{l}{\mu r^2}}{\dot{r}} dr \quad \text{—— } (**)$$

$\dot{r}$ → can be obtained from eqn (*)

Integrating (**)

$F(r) \propto r^n$

$n = 1, -2, -3$

expressible in terms of sin, cos fns.

$$\boxed{\theta(r) = \int \frac{\pm (l/r^2)\, dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}} \quad \text{—— } (1)$$

Obs. Since l is const in time

$$l = mr^2\dot{\theta}$$

$\dot{\theta}$ cannot change sign.

$\theta(t)$ must monotonically increase or decrease with time.

Physics I

Lecture 20

Recall that a 1-d problem is in principle solvable completely

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r).$$

Using above to solve for $\dot{r}$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}} \quad \text{——} \; (*)$$

$$t = \pm\int \frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

$t = t(r)$

$\hookrightarrow$ invert to get $r(t)$

$\ell = mr^2\dot{\theta}$

$$\theta = \int \frac{\ell}{mr^2}\,dt$$

substitutes $r(t)$

Our interest is to find the trajectory $r(\theta)$.

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \qquad \left\{ \dot{\theta} = \frac{l}{\mu r^2} \right\}$$

$$= \frac{\frac{l}{\mu r^2}}{\dot{r}}dr \quad \text{—— } (**)$$

$\dot{r}$ → can be obtained from eqn (*)

Integrating (**)

$$\boxed{\theta(r) = \int \frac{\pm (l/r^2)\, dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}} \quad \text{—— } (1)$$

$F(r) \propto r^n$

$n = 1, -2, -3$

expressible in terms of sin, cos fns.

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

E-L eqn.

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \qquad \left\{ \theta : \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \right.$$

$$\Rightarrow \boxed{\mu r^2 \dot{\theta} = l} \quad \text{— (1')}$$

$$\boxed{\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U(r)}{\partial r} = F(r)} \quad \text{— (1)}$$

Change variable to $u = \frac{1}{r}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \quad \text{— (2)}$$

from (1') $\dot{\theta} = \frac{l}{\mu r^2} \longrightarrow \frac{du}{d\theta} = -\frac{1}{\cancel{r^2}}\frac{\mu \cancel{r^2}}{l}\dot{r} = -\frac{\mu}{l}\dot{r}$ — (3)

$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r} \quad — (3)$$

Now $$\frac{d^2u}{d\theta^2} = -\frac{\mu}{l}\frac{d\dot{r}}{d\theta} = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right) = -\frac{\mu}{l\dot{\theta}}\ddot{r}$$

again substitute for $\dot{\theta}$ above

$$\boxed{\frac{d^2u}{d\theta^2} = -\frac{\mu}{l\cdot l}\cdot\mu r^2\ddot{r} = -\frac{\mu^2}{l^2}r^2\ddot{r}} \quad — (4)$$

$\dot{\theta} = \frac{l}{\mu r^2}$

$\dot{\theta}^2 = \frac{l^2}{\mu^2 r^4}$

from (4) $$\boxed{\ddot{r} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2} \;;\; r\dot{\theta}^2 = \frac{l^2}{\mu^2}u^3} \quad — (5)$$

E-L eqn. in r

$$\mu(\ddot{r} - r\dot{\theta}^2) = F(r)$$

from (5) we have $\ddot{r} = -\frac{l^2}{\mu^2} u^2 \frac{d^2u}{d\theta^2}$

and $r\dot{\theta}^2 = \frac{l^2}{\mu^2} u^3$.

say

$$\begin{cases} F(r) = k/r^2 \\ F(1/u) = k u^2 \\ \text{R.H.S} = -\frac{\mu k}{l^2} \end{cases}$$

substitute in E-L eqn.

$$-\frac{l^2}{\mu} u^2 \frac{d^2u}{d\theta^2} - \frac{l^2}{\mu} u^2 = F\left(\frac{1}{u}\right).$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2} F\left(\frac{1}{u}\right)} \longrightarrow \text{path eqn.}$$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r)}$$

$r(\theta)$.

Notice that $l = 0$, eqn. blows up. But should we worry? $mr^2\dot{\theta} = 0$ $\quad \theta = \text{const}$, st line through origin

Qualitative analysis of motion

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) .$$

Recall 1d problem

$\dot{r} = 0$ gives turning pts.

$$E - U(r) - \frac{\ell^2}{2\mu r^2} = 0 .$$

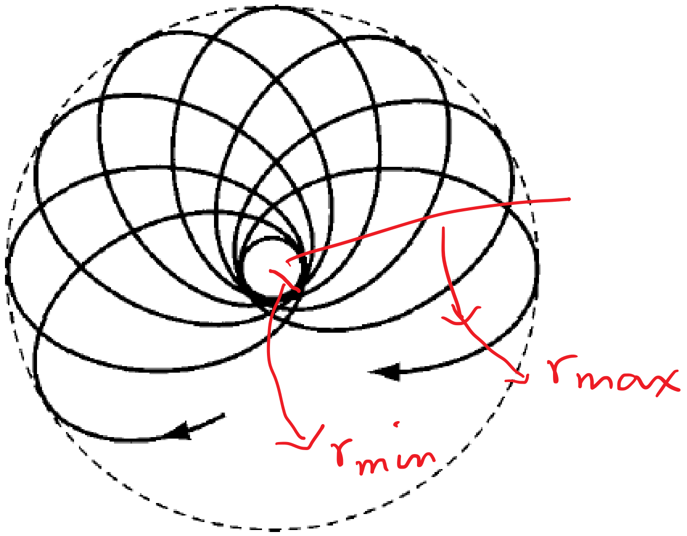
$E = U(x)$ turning pts.

$\downarrow$ motion is bounded & periodic.

$\downarrow$ In general possesses two roots r_{min}, r_{max}.

$$r_{min} \leq r \leq r_{max}$$

$\downarrow$ Can it be bounded but not periodic?



$r_{min} \leq r \leq r_{max}$

motion has to be confined to the annulus between r_{min} & r_{max}

If the motion is periodic, then orbit is closed

If the orbit does not close on itself after finite number of oscillations $\rightarrow$ open

Recall $\theta(r)$

$$\theta(r) = \int \frac{\pm (l/r^2)\, dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$$

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{l/r^2}{\sqrt{2\mu\left(E - U - l^2/2\mu r^2\right)}}$$

If $U(r) \propto r^{n+1}$

closed, non circular path can result only for $n = -2$ or $+1$

motion is symmetric in time

→ path is closed if $\Delta\theta$ is a rational fraction of 2π

$\Delta\theta = 2\pi \frac{m}{n}$, m, n are integers.

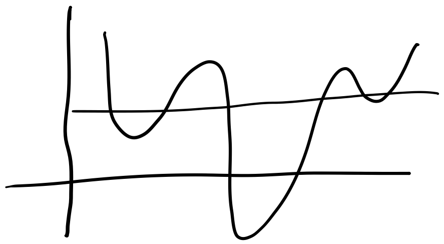
→ after n periods $\vec{r}$ made m complete revolutions.

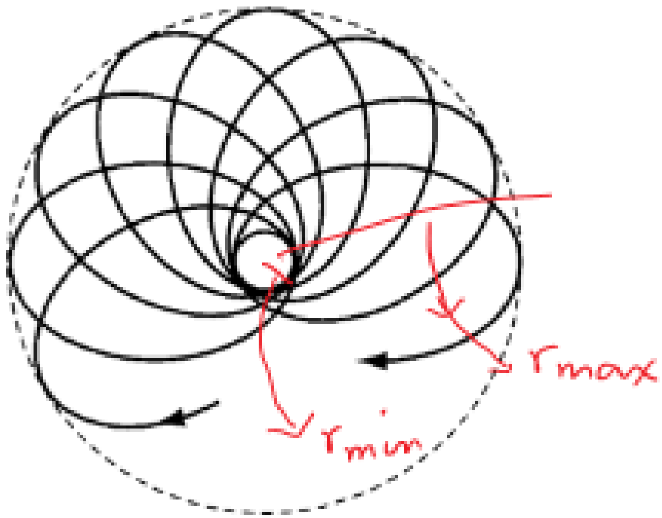
Physics I

Lecture 21

Recap

$E = U \rightarrow$ turning pts.





$r_{min} \leq r \leq r_{max}$

motion has to be confined to the annulus between r_{min} & r_{max}

If the motion is periodic, then orbit is closed $\rightarrow \Delta\theta = \frac{2\pi m}{n}$

If the orbit does not close on itself after finite number of oscillations $\rightarrow$ open

Effective potential

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{V(r) \,\equiv\, U_{eff}(r)}$$

$$\boxed{V(r) \equiv U(r) + \frac{\ell^2}{2\mu r^2}}$$

centrifugal potential energy

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r)$$

Let us specify $F(r) = -\frac{k}{r^2}$

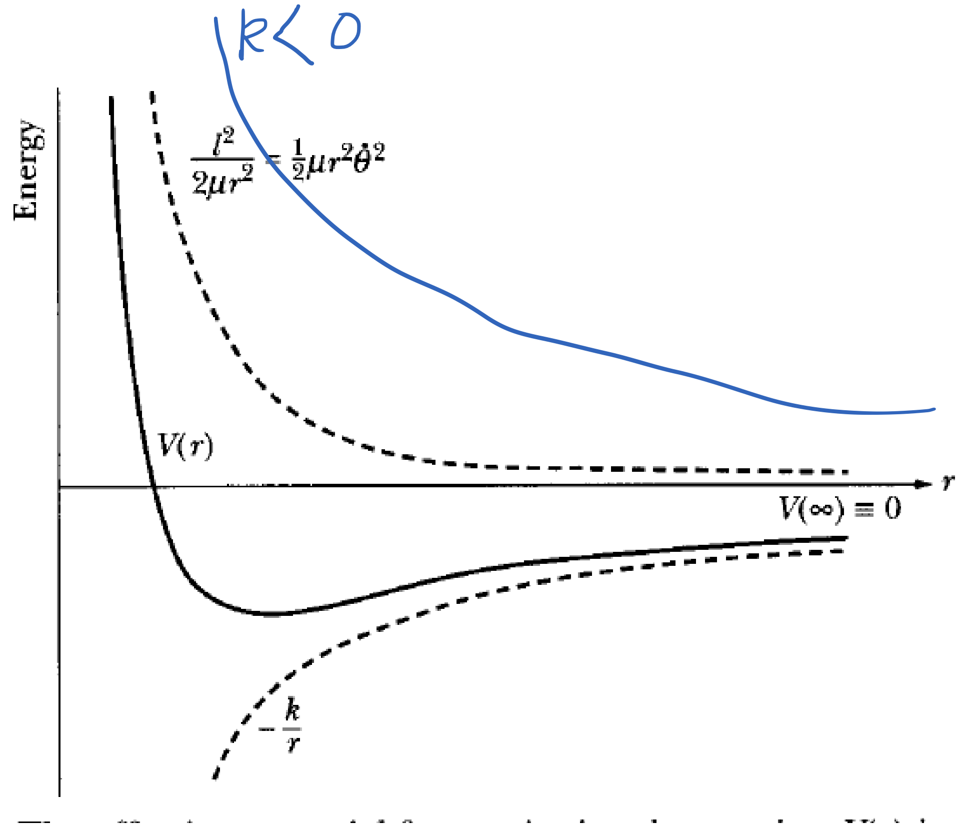
$U(r) = -\frac{k}{r}$ where we have taken $U(\infty) = 0$.

$$\boxed{V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}}$$

Recall : $E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$

But $\mu r^2\dot{\theta} = l$, $\dot{\theta} = \frac{l}{\mu r^2}$

$k < 0$



$$V(r) = \frac{l^2}{2\mu r^2} - \frac{k}{r}$$

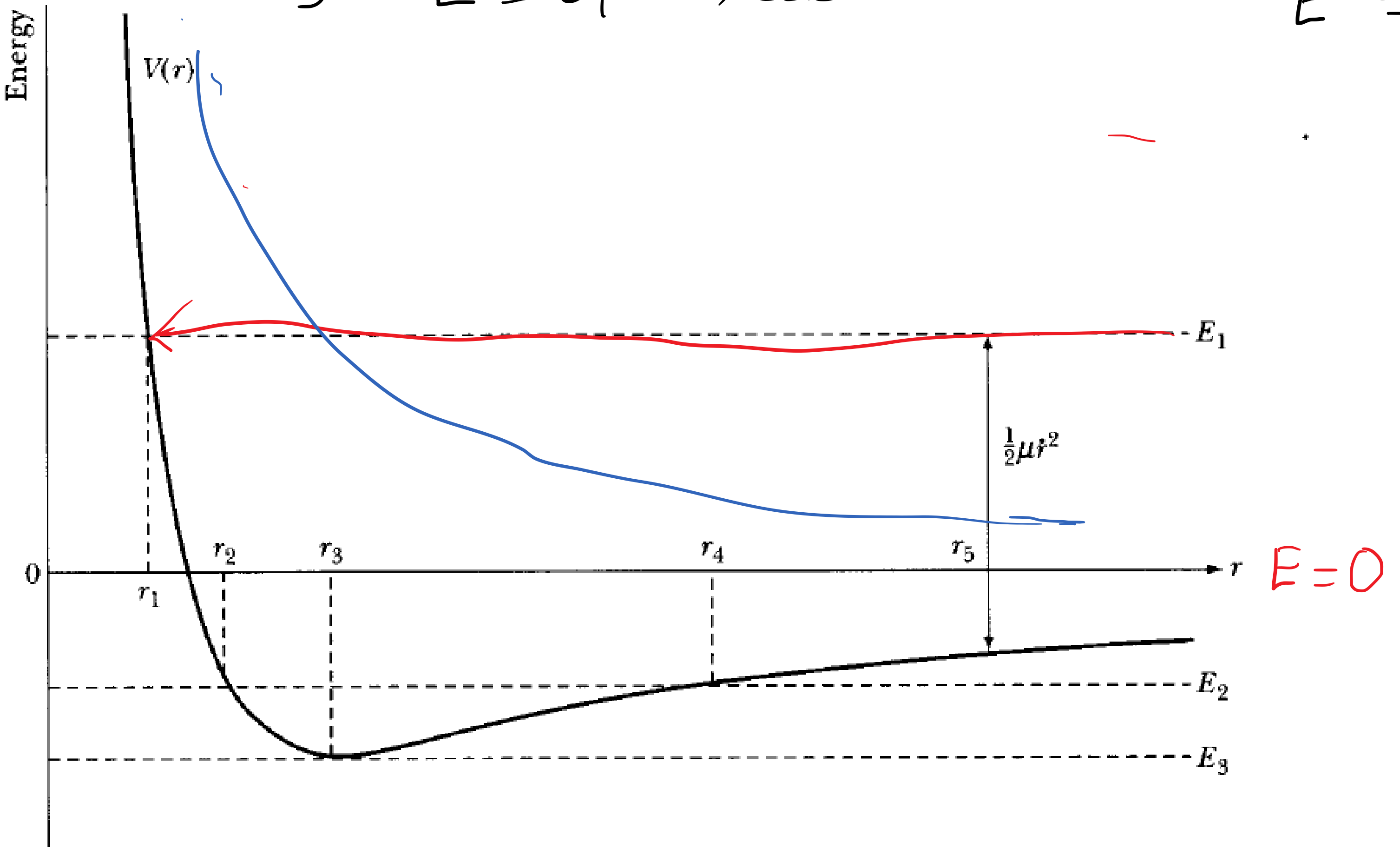
iv) $E = E_1 \longrightarrow$ also unbounded

$$E = \frac{1}{2}\mu \dot{r}^2 + V(r)$$

If $k < 0$

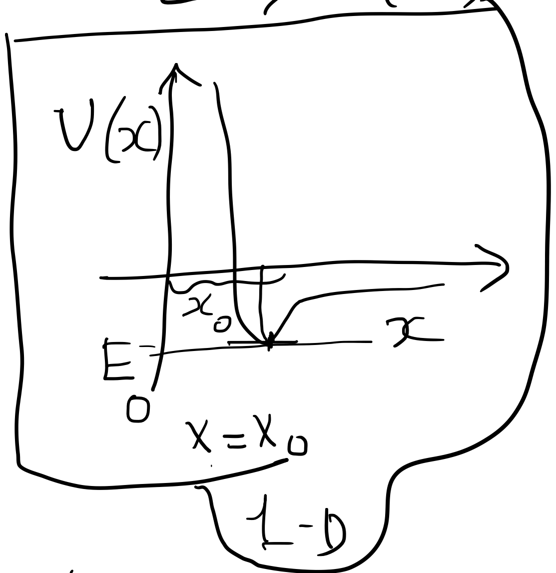
repulsive electrostatic potential

all energies unbounded motion



Classically allowed

$E > V(r)$



i) $E = E_3$, E_3 minimum , $r = r_3$, r = const, circular orbit.

ii) $E = E_2$, $r_2 \leq r \leq r_4$, bounded

iii) $E = 0$, unbounded, one turning pt

Recall the path eqn. [will give $r(\theta)$]

$$u = \frac{1}{r}.$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2u^2} F\left(\frac{1}{u}\right).$$

$$F = -\frac{k}{r^2}$$

$$= -ku^2.$$

$$= -\frac{\mu}{l^2u^2}\left(-ku^2\right)$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}} \longrightarrow$$ harmonic oscillator with const force

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}.$$

solving

$$\boxed{u = \frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_0)} \quad (1)$$

θ_0 gives the initial position θ, orientation of orbit in plane.

Let us take A to be positive, which can be always done

$$u = \frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_0) \quad \text{--- (1)}$$

Determine turning points from (1) $[r_1, r_2]$

$$\frac{1}{r_1} = \frac{\mu k}{l^2} + A \quad \text{--- (2)}$$

and $$\frac{1}{r_2} = \frac{\mu k}{l^2} - A \quad \text{--- (3)}$$

If we have $A > \frac{\mu k}{l^2}$, there will be only 1 turning pt.

r must be +ve.

We will compare the turning pts, with solns of $E = V$

$$E = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \longrightarrow$$ determines turning pts.

$$E = V(r) = U_{eff}.$$

$$\frac{l^2}{2\mu r^2} - \frac{k}{r} - E = 0$$

Solns are.

$$\frac{1}{r_1} = \frac{\mu k}{l^2} + \left[\left(\frac{\mu k}{l^2}\right)^2 + \frac{2\mu E}{l^2}\right]^{1/2} \quad (4)$$

$$\frac{1}{r_2} = \frac{\mu k}{l^2} - \left[\left(\frac{\mu k}{l^2}\right)^2 + \frac{2\mu E}{l^2}\right]^{1/2} \quad (5)$$

Comparing 2, 3 & 4, 5, can determine A.

$$A^2 = \frac{\mu^2 k^2}{l^4} + \frac{2\mu E}{l^2} \quad \text{— } 6$$

Recall soln:

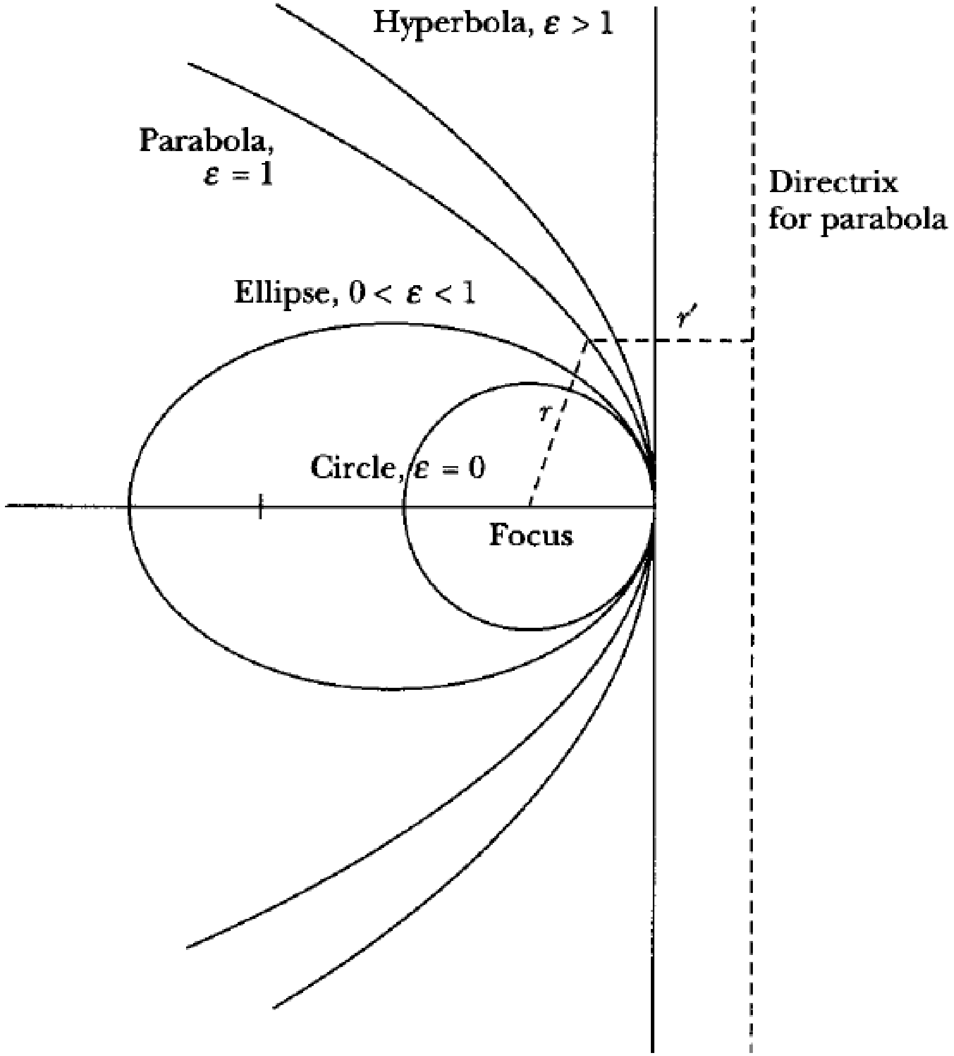
$$\frac{1}{r} = u = \frac{\mu k}{l^2} + A\cos\theta \quad \text{— } 1 \quad \text{Let } \theta_0 = 0$$

Let us define $\alpha \equiv \frac{l^2}{\mu k}$, $\varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$

Using these definitions, can rewrite 1

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta} \quad \text{— } 7$$

$\longrightarrow$ equation for general conic section in polar coordinates.

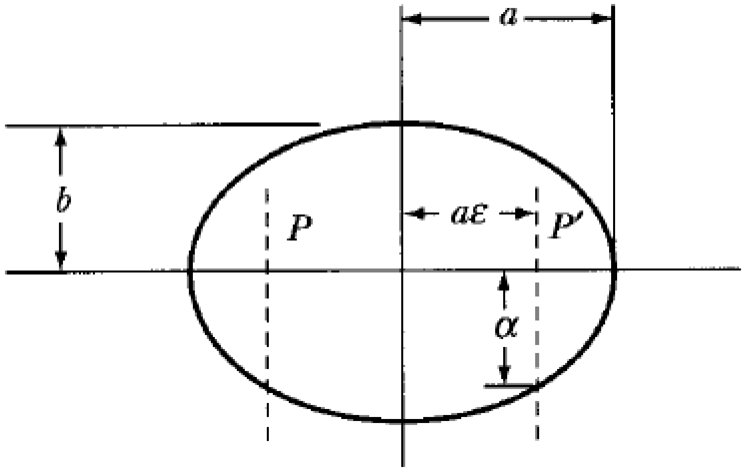


$\varepsilon > 1, E > 0$ hyperbola

$\varepsilon = 1 \quad E = 0$ parabola

$0 < \varepsilon < 1 \quad V_{min} < E < 0$
↳ ellipse

$\varepsilon = 0 \quad E = V_{min}$ circle



a
b
P
aε
P′
α

Physics I

Lecture 22

Recap

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

solved path equation

eccentricity

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

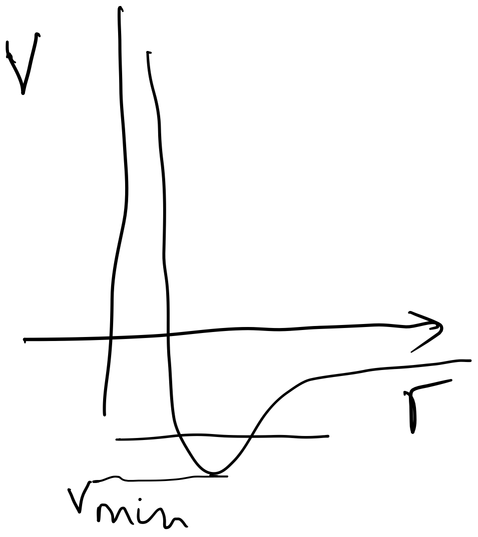
where $\alpha \equiv \frac{l^2}{\mu k}$

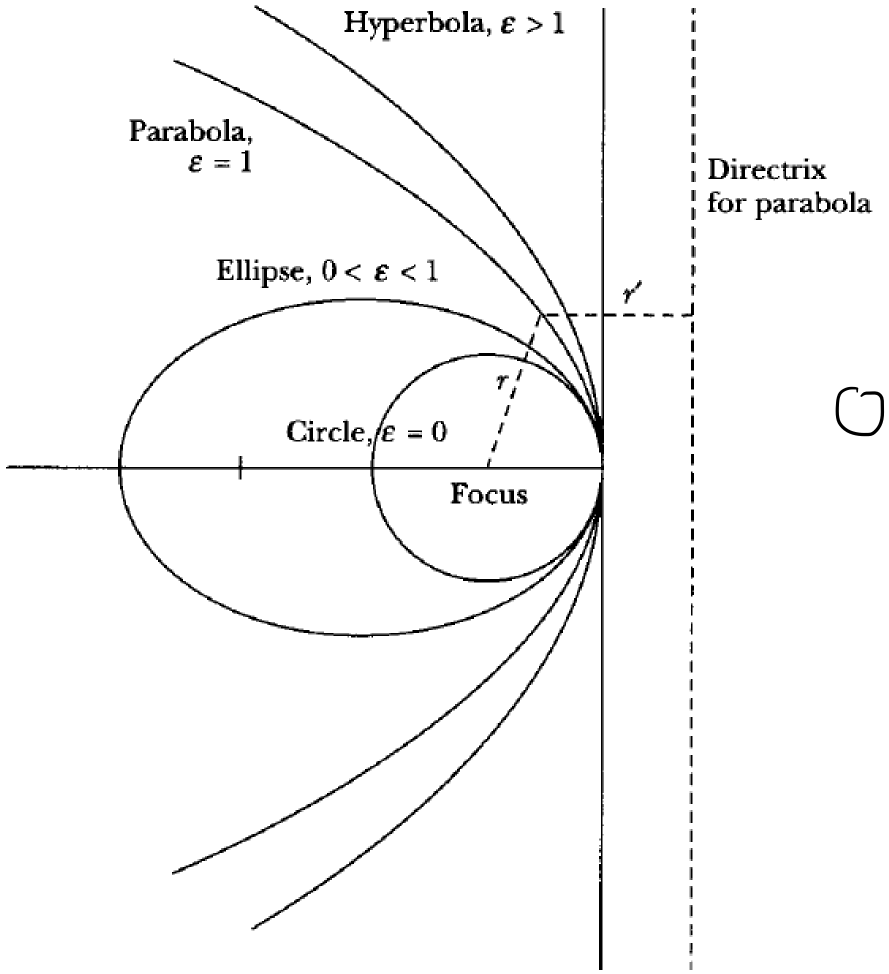
$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$.

conic section

familiar form recovered $\longrightarrow$ $x = r\cos\theta$

$y = r\sin\theta$.





$\varepsilon > 1 \quad E > 0$ hyperbola

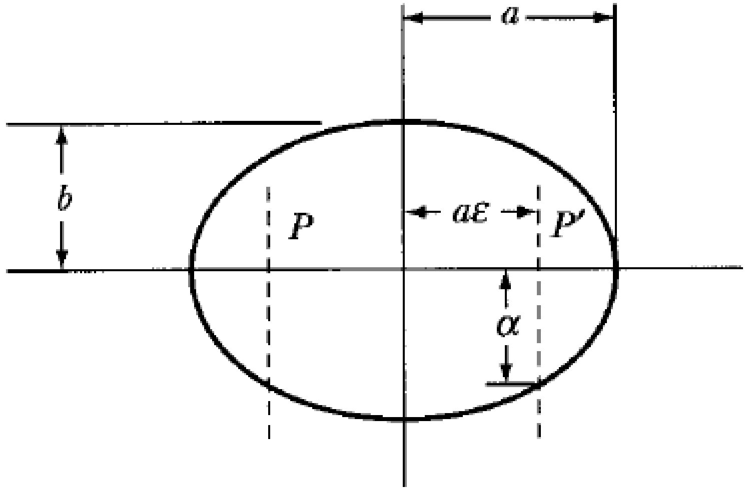
$\varepsilon = 1, \; E = 0$ parabola

$0 < \varepsilon < 1, \; V_{min} < E < 0$

↓ ellipse

$\varepsilon = 0, \; E = V_{min}$

circular.



$r_{min} = a(1-\varepsilon) = \dfrac{\alpha}{1+\varepsilon}$.

$r_{max} = a(1+\varepsilon) = \dfrac{\alpha}{1-\varepsilon}$

pt corresponding to closest approach
perihelion

For planetary motion

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

Recall

$$\frac{dA}{dt} = \frac{l}{2\mu}$$: { Rate of sweeping out area }

Entire area of ellipse $\equiv$ swept out in one time period.

$$\int_0^{\tau} dt = \frac{2\mu}{l} \int_0^{A} dA .$$

$$\tau = \frac{2\mu}{l} A = \frac{2\mu}{l} \pi a b = \frac{2\mu}{l} \pi \frac{k}{2|E|} \cdot \frac{l}{\sqrt{2\mu|E|}}$$

$$\boxed{\tau = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2}} \quad \text{—— } (*)$$

$$\tau = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} \quad \text{—} \; * $$

$$\left\{ a = \frac{k}{2|E|} \right\} .$$

$$|E| = \frac{k}{2a} .$$

$$b = \sqrt{\alpha}\, a \qquad \alpha = \frac{l^2}{\mu k} .$$

Squaring *

$$\tau^2 = \pi^2 k^2 \frac{\mu}{2} |E|^{-3} .$$

$$\boxed{\tau^2 = \frac{4\pi^2 \mu}{k} a^3}$$

Kepler's Third Law

with $m \to \mu$.

$$k = G m_1 m_2$$

$$\tau^2 = \frac{4\pi^2 \mu}{k} a^3 \qquad \left\{ \mu = \frac{m_1 m_2}{m_1 + m_2} \right\}$$

$$= \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

$$\boxed{\tau^2 \cong \frac{4\pi^2 a^3}{G m_2}} \quad m_1 \ll m_2 .$$

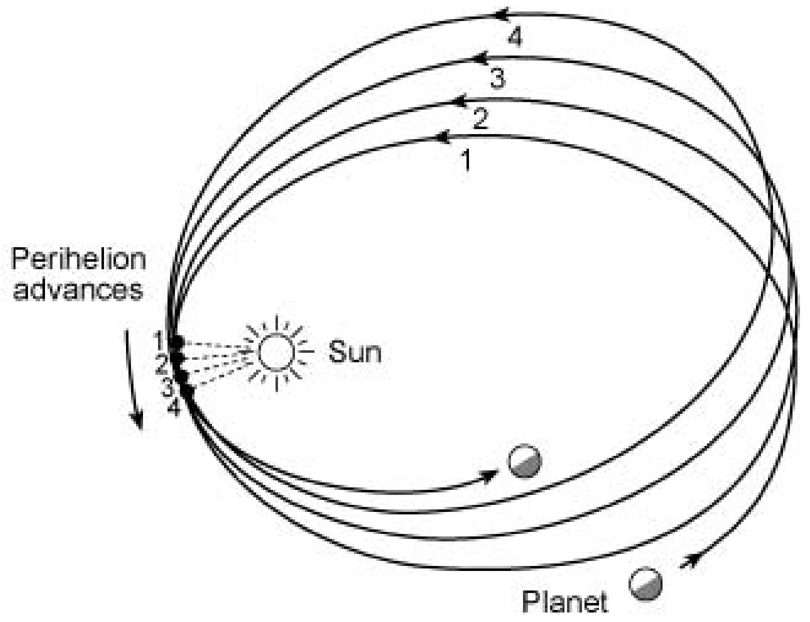
I. *Planets move in elliptical orbits about the Sun with the Sun at one focus.* ✓

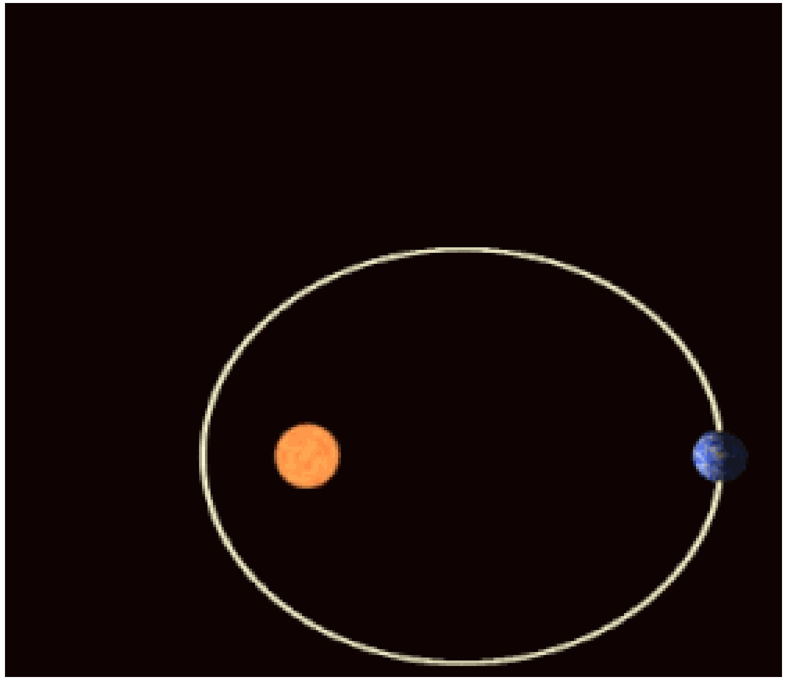
II. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.* ✓

III. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.* ✓

Actual orbits of planets are not strictly elliptical

→ nearby planets perturb the sun-planet gravitation field, ellipses dont come back to same point ⟶ perihelion precension


4
3
2
1
Perihelion advances
1
2
3
4
Sun
Planet



Mercury has largest perihelion shift
574'' arc sec / century

↳ all but 43'' / century could be explained by perturbations from other planets.

↳ this was explained by Einstein GR.

"effective correction ~~for~~ from GR" $\sim \frac{1}{r^4}$

Path eqn:

~~Solve perturbatitively~~

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)$$

here

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{l^2} + \frac{3GMu^2}{c^2}$$

small term

Conserved quantities in central force motion

E, l ~~α~~

Laplace – Runge – Lenz vector:

$\vec{F}(r) = -\frac{k}{r^2}\hat{r}$, $\vec{L}$: angular momentum

$E = \frac{1}{2}mv^2 - \frac{k}{r}$.

$$\boxed{\vec{A} = \vec{P} \times \vec{L} - mk\hat{r}}$$

$\hookrightarrow$ $\vec{A} \perp \vec{L}$; $\vec{P} \times \vec{L}$ and $\hat{r}$ are $\perp$ $\vec{L}$

$\downarrow$ $\vec{A}$ lies in plane of motion

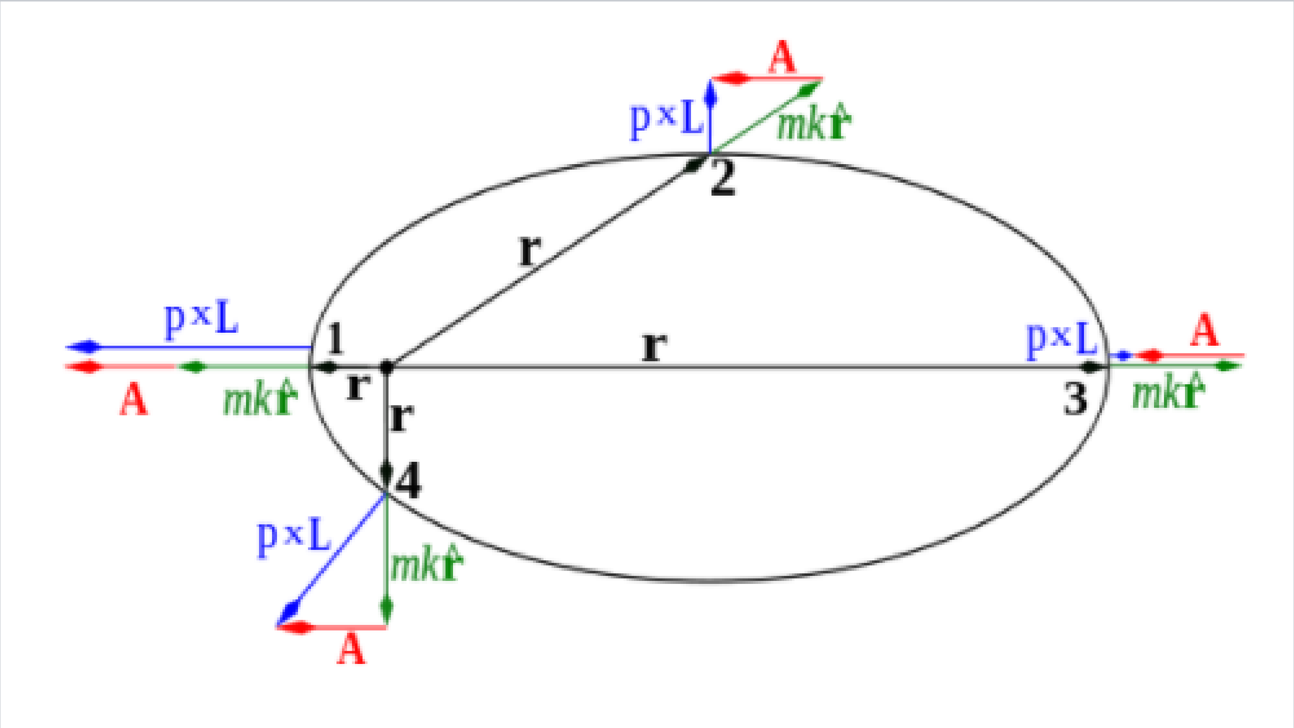


Figure 1: The LRL vector **A** (shown in red) at four points (labeled 1, 2, 3 and 4) on the elliptical orbit of a bound point particle moving under an inverse-square central force. The center of attraction is shown as a small black circle from which the position vectors (likewise black) emanate. The angular momentum vector **L** is perpendicular to the orbit. The coplanar vectors **p** × **L** and (*mk*/*r*)**r** are shown in blue and green, respectively; these variables are defined below. The vector **A** is constant in direction and magnitude.

Conservation

$$\vec{F} = \frac{d\vec{p}}{dt} = f(r)\frac{\vec{r}}{r} = f(r)\,\hat{r} \longrightarrow \text{central } f.$$

Want to show $\dfrac{d\vec{A}}{dt} = 0$ $\quad \dfrac{d}{dt}\left[\vec{p}\times\vec{L} - mk\hat{r}\right] = 0$

$$\frac{d}{dt}\left(\vec{p}\times\vec{L}\right) = \frac{d\vec{p}}{dt}\times\vec{L} \qquad \left(\because \frac{d\vec{L}}{dt} = 0\right).$$

$$= f(r)\,\hat{r}\times\left(\vec{r}\times m\frac{d\vec{r}}{dt}\right) \qquad \left[\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\right]$$

$$= f(r)\frac{m}{r}\left[\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) - r^2\frac{d\vec{r}}{dt}\right]$$

$$= f(r)\,\frac{m}{r}\left[\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) - r^2\frac{d\vec{r}}{dt}\right]$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot\frac{d\vec{r}}{dt} = \frac{d}{dt}(r^2) = 2r\frac{dr}{dt}$$

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -mf(r)\,r^2\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right]$$

$$= -mf(r)\,r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = -mf(r)\,r^2\frac{d(\hat{r})}{dt}.$$

Now $f(r) = -\dfrac{k}{r^2}$

$$= \frac{mk}{\cancel{r^2}}\cdot\cancel{r^2}\frac{d\hat{r}}{dt} = \frac{d}{dt}(mk\hat{r}).$$

$$\frac{d}{dt}(\vec{p} \times \vec{L}) = \frac{d}{dt}(mk\hat{r})$$

$$\boxed{\frac{d\vec{A}}{dt} = \frac{d}{dt}(\vec{p} \times \vec{L}) - \frac{d}{dt}(mk\hat{r}) = 0}$$

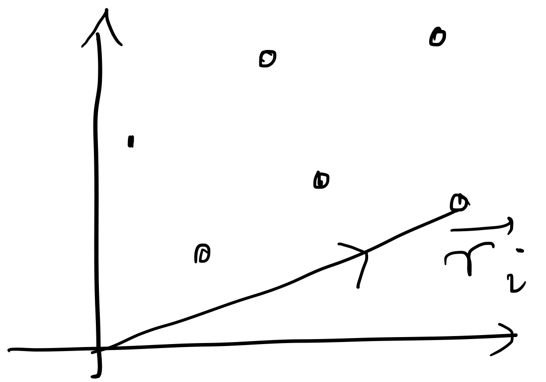
$$\frac{d\vec{A}}{dt} = 0$$

Physics I

Lecture 23

Many particle system dynamics

N particles



Newton's Law i^{th} particle

$$\dot{\vec{p}}_i = \sum_{\substack{j,\\ i\neq j}} \vec{F}_{ji} + \vec{F}_i^{(e)} \quad \text{--- (1)}$$

Assume Newton's 3rd Law (weak form)

$$\vec{F}_{ij} = -\vec{F}_{ji} \quad \text{--- (2)}$$

not necessarily acting along line joining particles

Sum (1) over all particles.

$$\sum_{i=1}^{N} \dot{\vec{p}}_i = \underbrace{\sum_i \sum_{\substack{j\\ i\neq j}} \vec{F}_{ji}}_{=0} + \underbrace{\sum_i \vec{F}_i^{(e)}}_{\vec{F}^{(e)}} \quad \text{--- (3)}$$

$\vec{F}^{(e)}$ = total ext force

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i$$

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)} \quad \text{--- (3)}$$

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \vec{F}^{(e)} \quad \text{--- (4)}$$

centre of mass coordinate.

Define $\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$ --- (5)

(4) reduces to

$$\boxed{M \frac{d^2\vec{R}}{dt^2} = \vec{F}^{(e)}} \quad \text{--- (6)}$$

purely internal forces have no effect on motion of CM.

Total linear momentum

$$\vec{P} = \sum_i m_i \frac{d\vec{r}_i}{dt} = M \frac{d\vec{R}}{dt} \quad \text{--- (7)}$$

$$\boxed{\text{If } \vec{F}^{(e)} = 0 \text{, total linear momentum is conserved}}$$

Angular Momentum

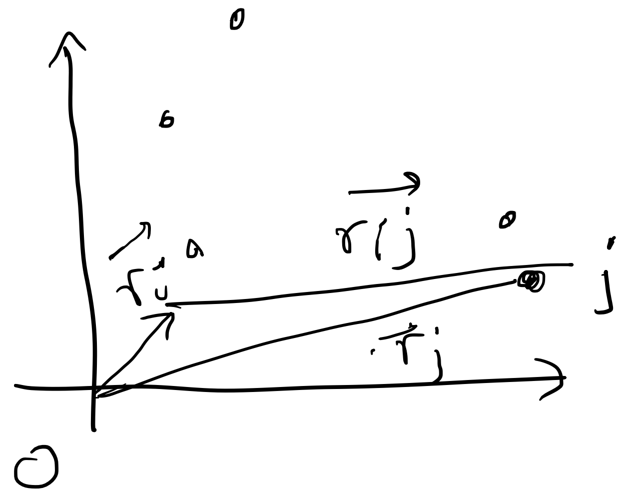
$$\vec{L}_{tot} = \sum_i (\vec{r}_i \times \vec{p}_i) = \vec{L}$$

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times F_i^{(e)} + \sum_i \sum_{\substack{j \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji} \quad \text{--- (8)}$$

Last term can be considered as sum of pairs of the following form

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}}_{\text{vanishes}} \quad \text{--- (9)}$$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ in the direction of $\vec{F}_{ji}$



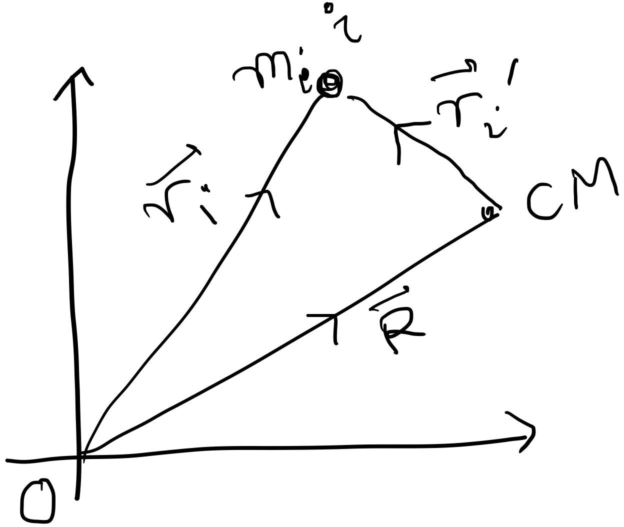
So if strong form of 3^{rd} Law holds

$$(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = 0$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}^{(e)} = \vec{N}^{(e)}} \quad (10)$$

external torque

$$\boxed{\vec{N}^{(e)} = 0 \implies \vec{L} \text{ is conserved}}$$



Angular momentum about origin

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$\boxed{\begin{aligned} \vec{V} &= \dot{\vec{R}} \\ \vec{v}_i{}' &= \dot{\vec{r}}_i{}' \\ \vec{v}_i &= \dot{\vec{r}}_i \end{aligned}}$$

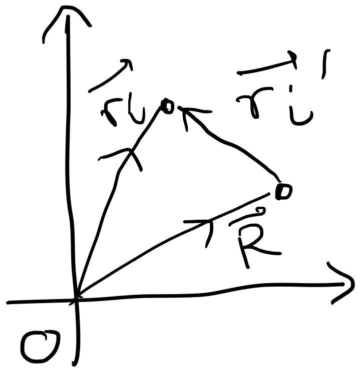
$$\left.\begin{aligned} \vec{r}_i &= \vec{r}_i{}' + \vec{R} \\ \vec{v}_i &= \vec{v}_i{}' + \vec{V} \end{aligned}\right\} \text{— (11)}$$

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{v}_i + \sum_i \vec{r}_i{}' \times m_i \vec{v}_i{}'$$

$$+ \sum_i \vec{r}_i{}' \times m_i \vec{V} + \sum_i \vec{R} \times m_i \vec{v}_i{}' \quad \text{— (12)}$$

Rewriting (12)

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r_i}' \times m_i \vec{v_i}'$$



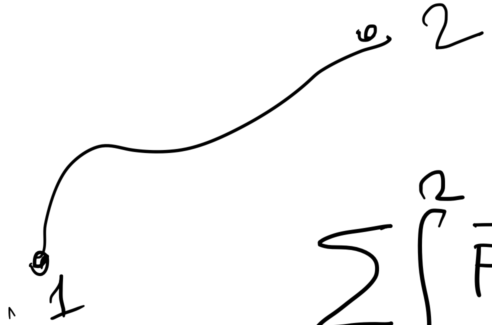
$$+ \underbrace{\left(\sum_i m_i \vec{r_i}'\right)}_{=0} \times \vec{V} + \vec{R} \times \frac{d}{dt}\underbrace{\left(\sum_i m_i \vec{r_i}'\right)}_{=0}.$$

$$\boxed{\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r_i}' \times \vec{p_i}'} \quad (13)$$

If the C.M is at rest w.r.t O, angular momentum will be independent of point of ref.

Energy

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i$$



$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{\substack{j \\ i \neq j}} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i$$

Using eqns of motion

$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 m \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \sum_i \int_1^2 d\left(\frac{1}{2} m v_i^2\right)$$

$$W_{12} = T_2 - T_1, \quad \text{where } T = \frac{1}{2} \sum_i m_i v_i^2$$

Making use of transfn. to CM coordinates.

$$T = \frac{1}{2} \sum_i m_i (\vec{v}_i' + \vec{V}) \cdot (\vec{v}_i' + \vec{V}).$$

$$= \frac{1}{2} \sum_i m_i V^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \vec{V} \cdot \frac{d}{dt}\underbrace{\left(\sum m_i \vec{r}_i'\right)}_{=0}$$

$$\boxed{T = \underbrace{\frac{1}{2} M V^2}_{\text{K.E of CM}} + \underbrace{\frac{1}{2} \sum_i m_i v_i'^2}_{\text{K.E of motion about the C.M.}}} \quad (13)$$

RHS -

$$\sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i$$

If ext force conservative.

$$= \sum_i \int_1^2 -\vec{\nabla}_i U_i \cdot d\vec{s}_i = -\sum_i \int_1^2 dU_i = -\sum_i U_i \Big|_1^2$$

If internal forces also conservative.

$\vec{F}_{ij}$ can be derived from a potential U_{ij}

$U_{ij} = U_{ij}(|\vec{r}_i - \vec{r}_j|) \longrightarrow$ to satisfy 3rd Law

$$\vec{F}_{ij} = -\vec{\nabla}_i U_{ij} = +\nabla_j U_{ij} = -\vec{F}_{ji} \quad (14)$$

$$-\vec{F}_{ij} = \vec{\nabla}_i \, U_{ij}(\vec{r}_i - \vec{r}_j) = (\vec{r}_i - \vec{r}_j) f$$

f → scalar fn.

$(\vec{r}_i - \vec{r}_j)$ → in the direction of line joining two particles.

Please fill in steps.

↓

finally, we find that

Total potential energy

$$\boxed{U = \sum_i U_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} U_{ij}}$$

Consequence → $\boxed{T + U \Rightarrow \text{conserved}}$

Physics I

Lecture 24

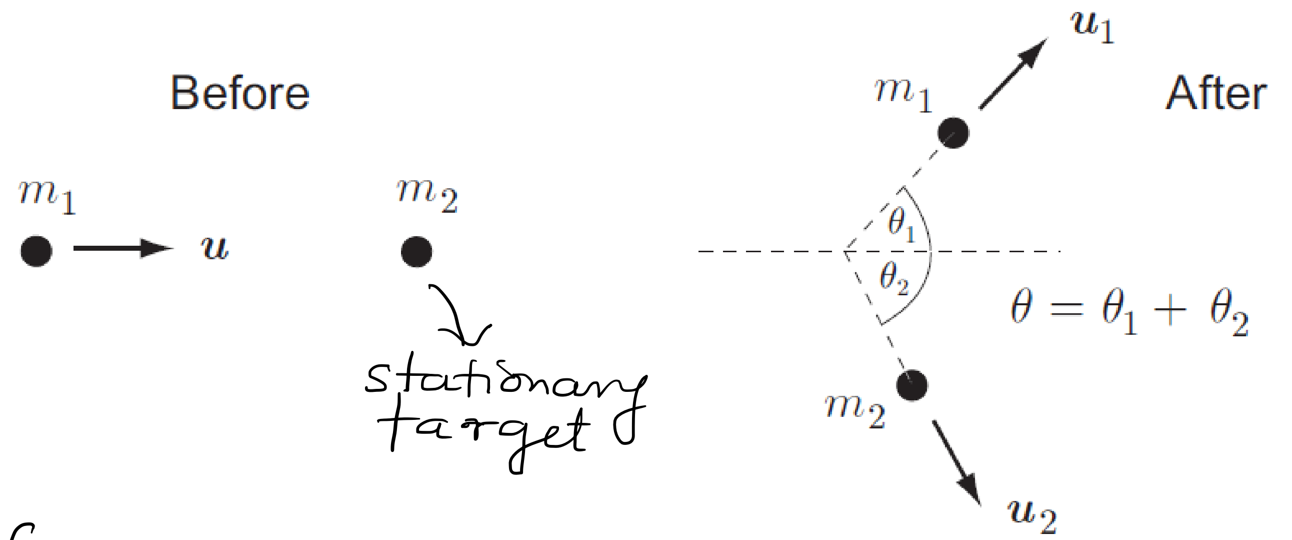
Looked at $\vec{P}$, $\vec{L}$ and E conservation for many particles

Today we will analyze collisions.

<u>Collision processes</u>

↳ Suppose that the mutual interactions between two particles $\longrightarrow 0$ as distance between them $\longrightarrow \infty$. so far apart each moves with constant velocity

ex: collison of balls, Rutherford scattering



LAB frame, θ_1 = scattering angle
θ_2 = recoil angle.

Linear momentum is conserved

$$\boxed{m\vec{u} = m_1\vec{u_1} + m_2\vec{u_2}} \quad \text{--- (1)}$$

$\Downarrow$ linear reln between $\vec{u}, \vec{u_1}, \vec{u_2} \Rightarrow$ 3 velocities lie in a plane
2D problem

Collisions are not kinetic energy preserving in general

Cons. of energy

$$\boxed{\frac{1}{2} m_1 u^2 + Q = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2} \quad \text{— (2)}$$

Q → energy gained (lost) in collision

Elastic collisions ⟹ K.E conserved.

↓ Is the notion of "elastic collisions" frame invariant?

Recall that we proved that $\left[v_i' = \vec{v}_i - \vec{V} \right]$

$$T = \underbrace{T^{CM}}_{\underbrace{\frac{1}{2}MV^2}_{\Downarrow}} + \underbrace{T^{G}}$$

T^G : $\frac{1}{2}\sum_i m_i v_i'^2$. K.E of motion about the CM.

T^G : frame independent.

$\frac{1}{2}MV^2$: preserved in collision.

(not affected by mutual interaction)

Hence the notion of "elastic collision" is indeed frame independent

Elastic Collisions

$$\frac{1}{2} m_1 u^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \quad \text{--- (3)}$$

mom. cons.

$$m_1 \vec{u} = m_1 \vec{u_1} + m_2 \vec{u_2} \quad \text{--- (1)}$$

Take scalar product of each side of (1)

$$m_1^2 u^2 = m_1^2 u_1^2 + m_2^2 u_2^2 + 2 m_1 m_2 \vec{u_1} \cdot \vec{u_2} \quad \text{--- (4)}$$

eliminate u^2 between (4) & (3)

$$2 m_1 \vec{u_1} \cdot \vec{u_2} = (m_1 - m_2) u_2^2 \quad \text{--- (5)}$$

$$2m_1 \vec{u_1} \cdot \vec{u_2} = (m_1 - m_2) u_2^2$$

$$2m_1 u_1 u_2 \cos\theta = (m_1 - m_2) u_2^2$$

$$\boxed{\cos\theta = \frac{(m_1 - m_2)\, u_2}{2 m_1 u_1}} \quad (6)$$

provided $u_1 \neq 0$.

θ = opening angle $= \theta_1 + \theta_2$

Ex: Ball of mass m energy E in an elastic collision of mass 4m, initially at rest.

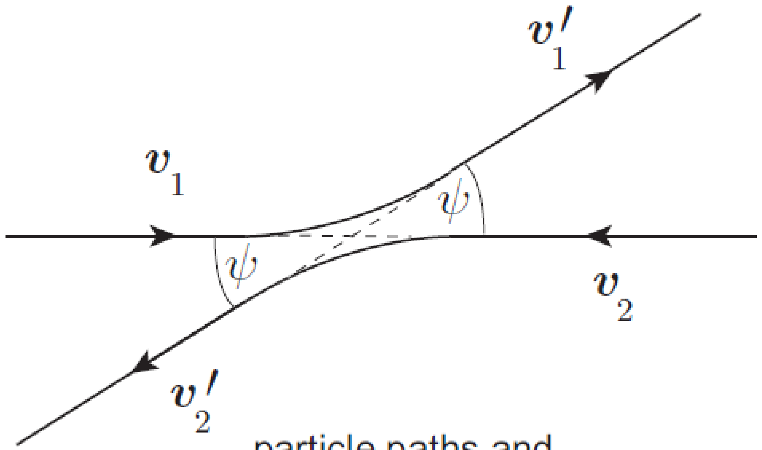
Observed; Two balls depart at 120° to each other.

$\frac{E_1}{E_2} = ?$ E_1, E_2 final energies

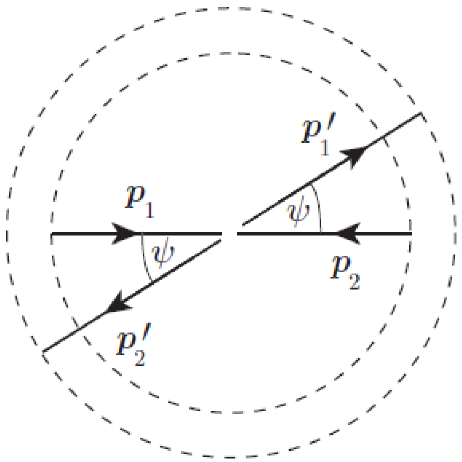
$$\cos\theta = \frac{(m_1 - m_2)\,u_2}{2 m_1 u_1} \Longrightarrow \boxed{\frac{u_1}{u_2} = 3}$$

$$\frac{E_1}{E_2} = \frac{\frac{1}{2} m\, u_1^2}{\frac{1}{2}\, 4m\, u_2^2} = \frac{9}{4}$$

Collision process in CM/ZM frame $\longrightarrow$ CM is at rest.



particle paths and initial and final velocities



initial and final momenta

Two particles, isolated system, CM, moves with constant velocity. So the frame in which CM $\equiv$ G at rest is an inertial frame.

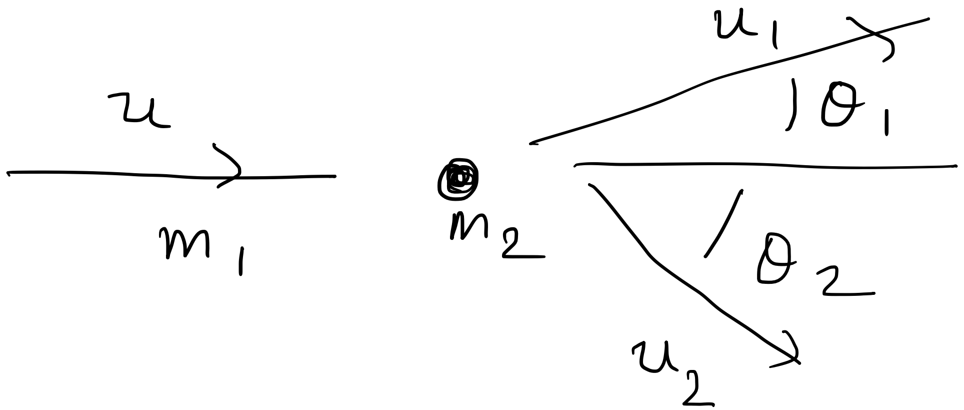
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \vec{V} = \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2} = 0$$

$$\Rightarrow \boxed{\vec{p}_1 + \vec{p}_2 = 0} \rightarrow \text{Zero momentum frame.}$$

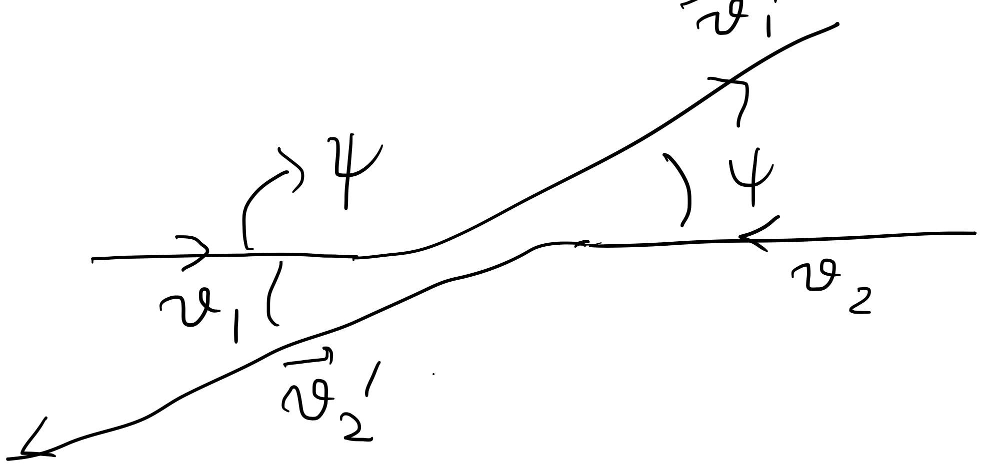
before $\quad \vec{p_1} + \vec{p_2} = 0$

after $\quad \vec{p_1}' + \vec{p_2}' = 0$

$\Longrightarrow$ ZM frame.



LAB



ZM.

$\longrightarrow$ each particle deflected through SAME angle ψ

total linear mom $\quad \vec{P} = m_1 \vec{u}$.

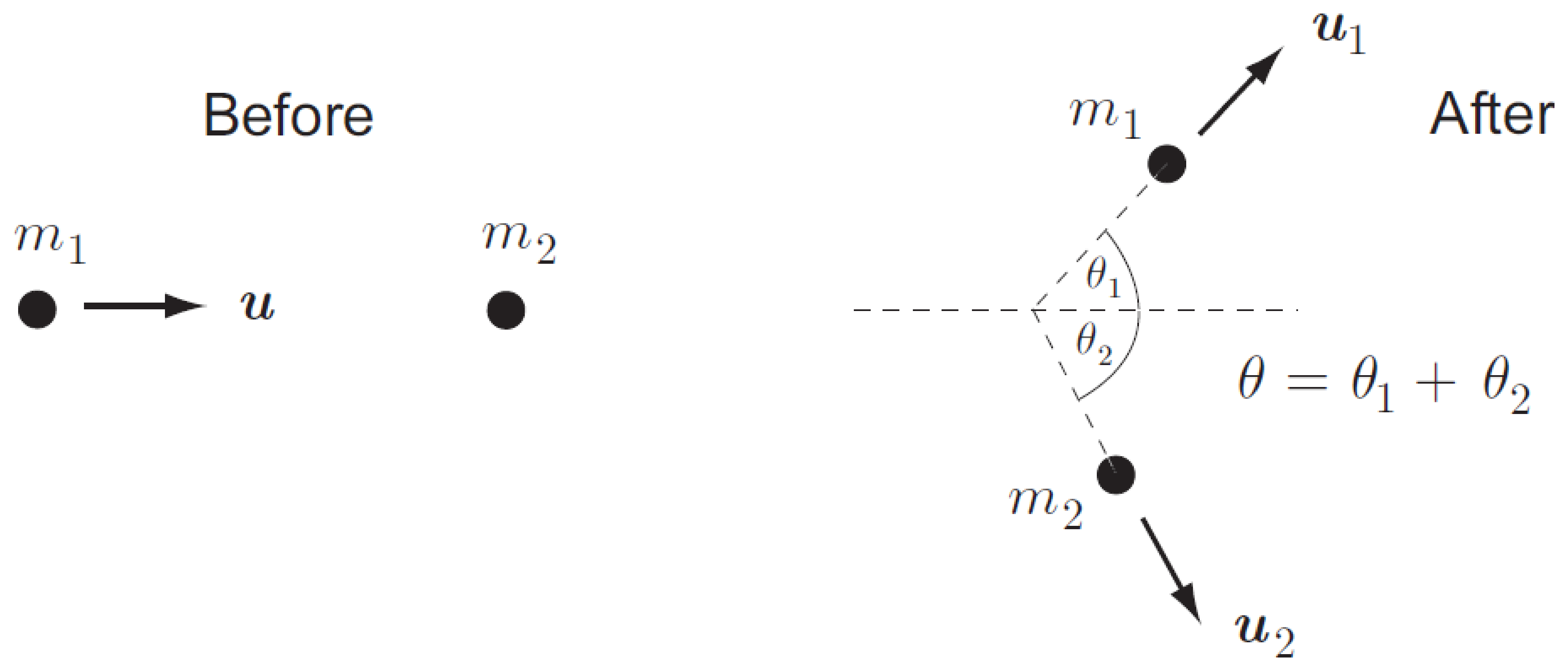
$$\vec{V} = \frac{m_1 \vec{u}}{m_1 + m_2}$$

vel of CM

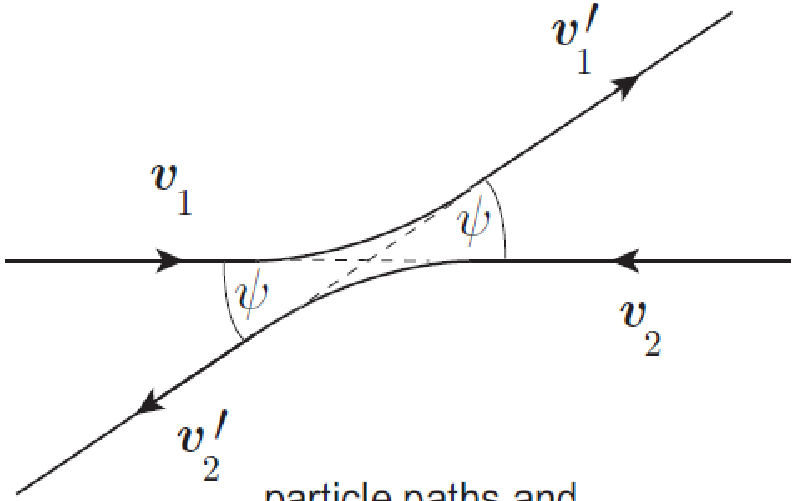
Physics I

Lecture 25

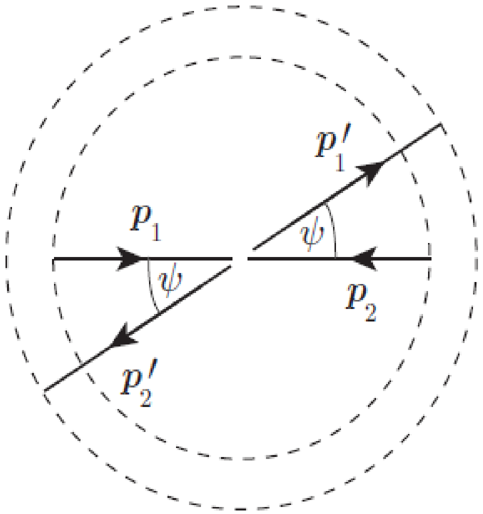
Lab frame


Before
m_1
$\boldsymbol{u}$
m_2
After
$\boldsymbol{u}_1$
m_1
θ_1
θ_2
$\theta = \theta_1 + \theta_2$
m_2
$\boldsymbol{u}_2$

ZM frame



particle paths and
initial and final velocities



initial and final momenta

$$\vec{V} = \frac{m_1 \vec{u}}{m_1 + m_2}$$

vel. of CM rel to lab frame.

Lab frame $\vec{P} = m_1 \vec{u} = (m_1 + m_2)\vec{V}$

$\vec{p}_1 = m_1 \vec{v}_1$, $\vec{p}_2 = m_2 \vec{v}_2$, $\vec{p}_1{}' = m_1 \vec{v}_1{}'$, $\vec{p}_2' = m_2 \vec{v}_2{}'$

$$\boxed{\vec{p}_1 + \vec{p}_2 = 0 \; ; \; \vec{p}_1{}' + \vec{p}_2{}' = 0}$$

each particle deflected through SAME angle ψ.

Conservation of energy

$$\frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 + Q = \frac{1}{2} m_1 |\vec{v}_1{}'|^2 + \frac{1}{2} m_2 |\vec{v}_2{}'|^2$$

Let p be the magnitude of initial common momentum

Let p' " " " " final " "

Cons. of energy

$$\boxed{\frac{p^2}{2m_1} + \frac{p^2}{2m_2} + Q = \frac{p'^2}{2m_1} + \frac{p'^2}{2m_2}}$$

for elastic collisions $Q = 0$, $p = p'$

(p, ψ) determine final momenta $\vec{p}_1, \vec{p}_2$.

$$\frac{p^2}{2m_1} + \frac{p^2}{2m_2} + Q = \frac{p'^2}{2m_1} + \frac{p'^2}{2m_2}$$

$$\boxed{p'^2 = p^2 + \left(\frac{2Q\, m_1 m_2}{m_1 + m_2}\right)}$$

$$Q = 0 \Rightarrow p'^2 = p^2 \;,\; p' = p.$$

In a typical scattering problem what is known are masses m_1, m_2 and initial $\vec{p_1}, \vec{p_2}$.

$$\boxed{\begin{aligned} \vec{v_1} &= \vec{u} - \vec{V} \\ \vec{v_2} &= -\vec{V} \end{aligned}}$$

$\rightarrow$ connection between Lab & ZM initial velocities.

$$\vec{V} = \frac{m_1 \vec{u}}{m_1 + m_2}.$$

Initial momentum in ZM frame.

$$p = m_2 v_2$$

noting that $|\vec{v_2}| = |\vec{V}|$.

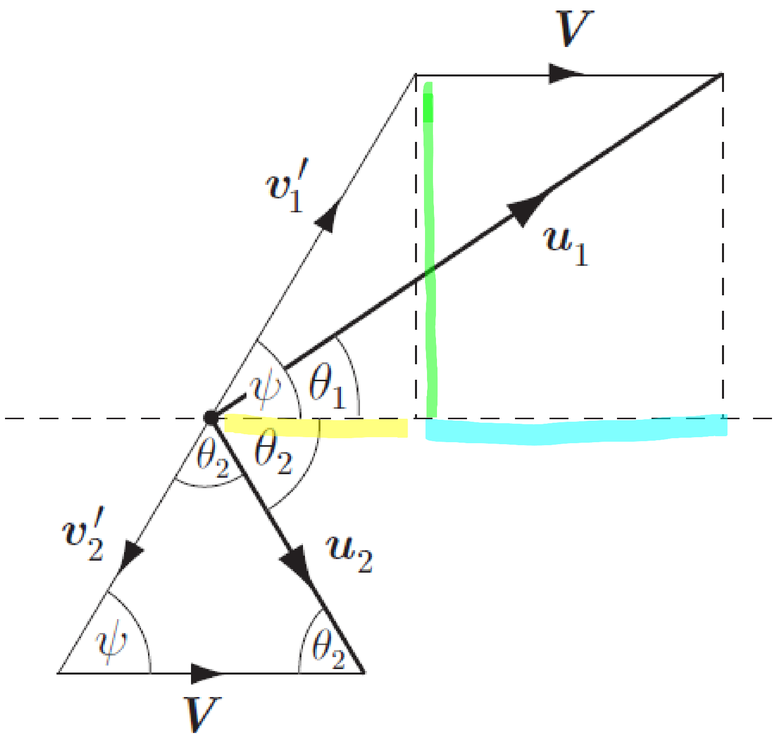
$$\boxed{p = \frac{m_1 m_2 u}{m_1 + m_2}}$$

$$p = m_2 \vec{v_2} = m_2 \vec{v_2}'.$$

Returning to Lab frame

elastic collisions . $Q = 0$.

$P' = p$.



$$\left.\begin{aligned} \vec{u}_1 &= \vec{v}_1' + \vec{V} . \\ \vec{u}_2 &= \vec{v}_2' + \vec{V} \end{aligned}\right\}$$

$$v_1' = \frac{m_2 u}{m_1+m_2} \ ; \quad v_2' = \frac{m_1 u}{m_1+m_2} = V . \ (*)$$

$$\tan\theta_1 = \frac{v_1' \sin\psi}{v_1'\cos\psi + V} = \frac{\sin\psi}{\cos\psi + V/v_1'}$$

$$\boxed{\theta_2 = \frac{1}{2}(\pi - \psi)}$$

$$\boxed{\tan\theta_1 = \frac{\sin\psi}{\cos\psi + m_1/m_2}}$$

opening angle $\theta = \theta_1 + \theta_2$

$$\tan\theta = \tan(\theta_1+\theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1-\tan\theta_1\tan\theta_2} = \left(\frac{m_1+m_2}{m_1-m_2}\right)\cot\frac{\psi}{2}$$

↳ supplement intermediate steps

To find the final energies.

$$\vec{u}_2 = \vec{v}_2{}' + \vec{V}$$

$$u_2^2 = v_2^2 + V^2 + 2\vec{v}_2{}'\cdot\vec{V}$$

$$= 2V^2 - 2V^2\cos\psi$$

$$= 4V^2\sin^2\psi/2$$

$$\boxed{u_2 = 2V\sin\frac{\psi}{2}}$$

Final energies

$$\frac{E_2}{E_0} = \frac{\frac{1}{2} m_2 u_2^2}{\frac{1}{2} m_1 u^2} = \frac{\frac{1}{2} m_2 \left(2V \sin \psi/2\right)^2}{\frac{1}{2} m_1 u^2}.$$

$$\boxed{\frac{E_2}{E_0} = \frac{4\, m_1/m_2}{(m_1/m_2 + 1)^2} \sin^2 \frac{\psi}{2}}.$$

Recall $V = \dfrac{m_1 u}{m_1 + m_2}$

$$\gamma = \frac{m_1}{m_2}$$

1. $$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \gamma}$$

2. $$\theta_2 = \frac{1}{2}(\pi - \psi).$$

3. $$\tan\theta = \left(\frac{\gamma+1}{\gamma-1}\right)\cot\frac{\psi}{2}$$

4. $$\frac{E_2}{E_0} = \frac{4\gamma}{(\gamma+1)^2}\sin^2\psi/2.$$

Physics I

Lecture 26

Quiz question

$\vec{F} = -k\vec{r}$. $\rightarrow U = \frac{1}{2}kr^2$

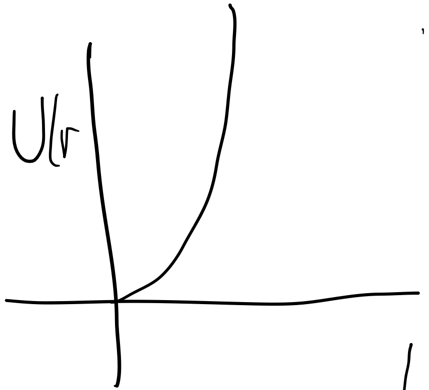
The particle can reach the origin.

$l = 0$

$mr^2\dot{\theta} = l = 0$

$\theta = \text{const}$.

meant to say can "always" reach the origin.



$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U(r)$$

$$\frac{1}{2}m\dot{r}^2 = E - \frac{l^2}{2mr^2} - U(r) > 0$$

$$\left(Er^2\right)_{r\to 0} - \frac{l^2}{2m} - \left(U(r)\,r^2\right)_{r\to 0} > 0$$

$$\left(U(r)\,r^2\right)_{r\to 0} < -\frac{l^2}{2m}$$

e.g.

$$U(r) = -\frac{\alpha}{r^n}$$

$n > 2$. $n = 2$, $\alpha < l^2/2m$

will reach centre

Elastic collision formulae

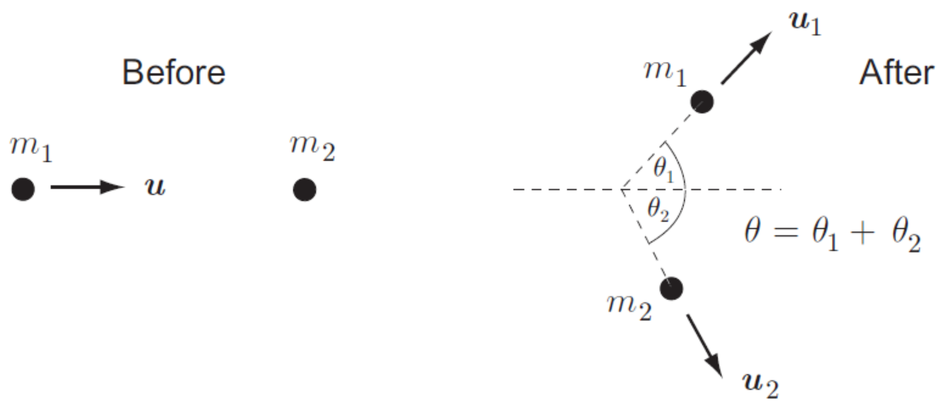
A. $\tan\theta_1 = \dfrac{\sin\psi}{\cos\psi + \gamma}$ B. $\theta_2 = \frac{1}{2}(\pi - \psi)$

C. $\tan\theta = \left(\dfrac{\gamma+1}{\gamma-1}\right)\cot(\tfrac{1}{2}\psi)$ D. $\dfrac{E_2}{E_0} = \dfrac{4\gamma}{(\gamma+1)^2}\sin^2(\tfrac{1}{2}\psi)$

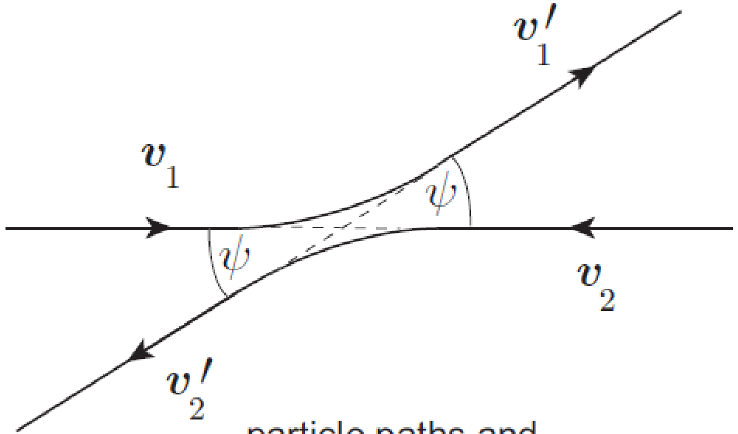
$$(10.22)$$

ψ is the scattering angle in the ZM frame, and $\gamma = m_1/m_2$, the mass ratio of the two particles.

LAB



ZM .



particle paths and initial and final velocities

In an experiment, particles of mass m and energy E are used to bombard stationary target particles of mass 2m. The experimenters wish to select particles that after scattering have an energy E/3. At which scattering angle will they find such particles ?

$E_2/E_0 = 2/3$.

$\theta_1 = ?$

$\frac{E_1}{E_0} = \frac{1}{3}$, $\frac{E_2}{E_0} = \frac{4\gamma}{(\gamma+1)^2} \sin^2 \psi/2$.

$\gamma = \frac{1}{2}$.

$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \gamma}$.

$\frac{2}{3} = \frac{4 \times \frac{1}{2}}{9} \times 4 \sin^2 \psi/2$

$\Rightarrow \infty$

$\boxed{\theta_1 = 90^\circ}$

$\sin^2 \psi/2 = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}}{\cancel{8}} \frac{3}{4}$; $\sin\frac{\psi}{2} = \frac{\sqrt{3}}{2}$.

$\psi = 120^\circ$.

In one collision, the opening angle was 45 degrees. What are the individual scattering angles ?

$r = \frac{1}{2}$, $\theta_1 = ?$, $\theta_2 = ?$.

$$\tan\theta = \left(\frac{r+1}{r-1}\right)\cot\psi/2 .$$

$$1 = \frac{\frac{1}{2}+1}{\frac{1}{2}-1}\cot\psi/2 \Rightarrow \boxed{\cot\frac{\psi}{2} = -\frac{1}{3}}$$

$$\psi/2 = 72^\circ$$

$$\theta_2 = \frac{1}{2}(\pi - \psi) = 90^\circ - 72^\circ = 18^\circ$$

$$\theta_1 = \theta - \theta_2 = 45^\circ - 18^\circ = 27^\circ$$

In another collision, the scattering angle was measured to be 45 degrees. What was the recoil angle ?

$$\gamma = \frac{1}{2}. \qquad \theta_1 = 45^\circ, \quad \theta_2 = ?$$

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + \gamma} = 1.$$

I made an error in the computation of θ_2 in class which, was corrected later by Pratichi Paromita. I am uploading the corrected version

$$\frac{\sin\psi}{\cos\psi + \frac{1}{2}} = 1 \Rightarrow 2\sin\psi - 2\cos\psi = 1$$

$$\sqrt{8}\sin(\psi - 45^\circ) = 1$$

$$\psi \cong 66^\circ, \Rightarrow \theta_2 = \frac{1}{2}(\pi - \psi) \simeq 56^\circ$$

In an elastic collision between an alpha particle and an unknown nucleus at rest the alpha particle was deflected through a right angle and lost 40% of its energy. Identify the mystery nucleus.

let the mass of unknown nucleus = M

$\gamma = \frac{4}{M}$

$M = 16$

→ Oxygen

Elastic collision formulae

A. $\tan\theta_1 = \dfrac{\sin\psi}{\cos\psi + \gamma}$

B. $\theta_2 = \frac{1}{2}(\pi - \psi)$

C. $\tan\theta = \left(\dfrac{\gamma+1}{\gamma-1}\right)\cot(\frac{1}{2}\psi)$

D. $\dfrac{E_2}{E_0} = \dfrac{4\gamma}{(\gamma+1)^2}\sin^2(\frac{1}{2}\psi)$

$$(10.22)$$

ψ is the scattering angle in the ZM frame, and $\gamma = m_1/m_2$, the mass ratio of the two particles.

$\frac{E_1}{E_0}$

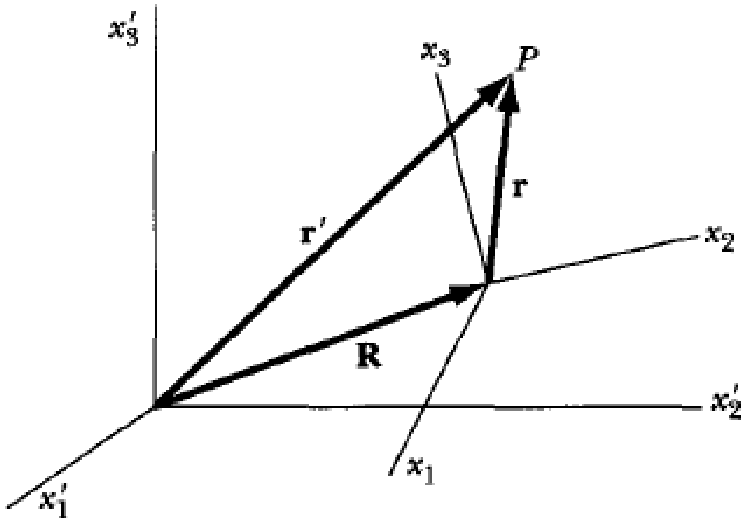
Motion in a Noninertial Reference Frame

- Newton's Laws are valid ***only*** in ***inertial*** frames
- However, there are problems where treating motion of the system in a non-inertial frames is simpler
- For example, to describe the motion of a body on earth, or near earth, it might be useful to use a coordinate system fixed on earth. This is clearly a non-inertial frame, since the earth rotates.
- To describe the motion of a rigid body which is free to rotate and accelerate, it is often convenient to use a reference frame fixed to the rigid body.

Physics I

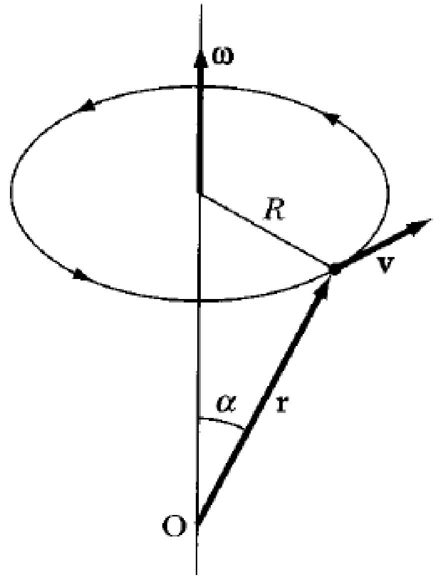
Lecture 27

Rotating Coordinate Systems

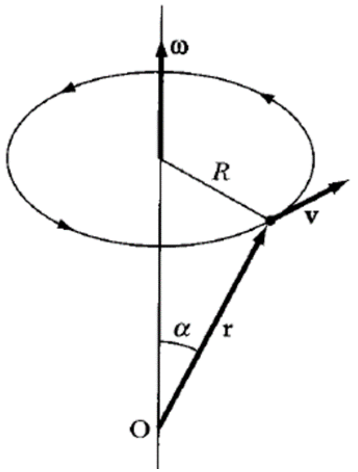


The x_i' are coordinates in the fixed system, and x_i are coordinates in the rotating system. The vector **R** locates the origin of the rotating system in the fixed system.

$$\boxed{\vec{r}\,' = \vec{R} + \vec{r}} \quad \text{— } (1)$$



Recall, we had learnt that a particle moving arbitrarily in space, can be considered , **at a given instant** to be moving in a **plane, circular path** about a given axis. An arbitrary infinitesimal displacement,(which can be a combination of translation and rotation) can always be represented by a “ pure rotation” about some axis called the instantaneous axis of rotation.
The line passing through the centre of the circle and perpendicular to the instantaneous direction of motion is called the instantaneous axis of rotation.



Rate of change of angular position = ω = angular velocity

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \quad \text{--- } (2)$$

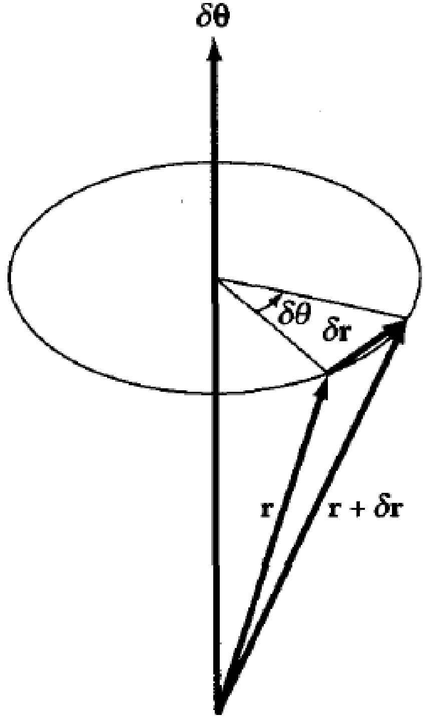
linear velocity $\vec{v} = \dot{\vec{r}}$

$$v = R\frac{d\theta}{dt} = R\omega \quad \text{--- } (3)$$

$\vec{v} \perp \vec{r}$

$$v = r\omega \sin\alpha \quad \text{--- } (5)$$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}} \quad \text{--- } (6)$$



$$\boxed{\delta\vec{r} = \delta\vec{\theta} \times \vec{r}} \quad \text{— (7)}$$

Getting back to our fixed vs rotating system

If x_i coordinate system undergoes infinitesimal rotation $\delta\theta$, for the motion P (at rest in x_i system)

$$\left(d\vec{r}\right)_{fixed} = d\vec{\theta} \times \vec{r} \quad \text{— (8)}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \frac{d\vec{\theta}}{dt} \times \vec{r} \quad \text{— } (9)$$

→ essentially same as (6)

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega} \times \vec{r} \quad \text{— } (10)$$

[P fixed in x_i system]

Now if the point P has velocity $\left(\frac{d\vec{r}}{dt}\right)_{rotating}$ w.r.t the x_i system, this must be added to $\vec{\omega} \times \vec{r}$ to obtain $\left(\frac{d\vec{r}}{dt}\right)_{fixed}$.

$$\boxed{\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{r}} \quad (11)$$

Allthough we have derived (11) w.r.t $\vec{r}$, i.e the displacement vector, this holds for any arbitrary vector $\vec{Q}$

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{Q}} \quad (12)$$

In particular $\vec{Q} = \vec{\omega}$ (13)

$$\left(\dot{\vec{\omega}}\right)_{fixed} = \left(\dot{\vec{\omega}}\right)_{rotating} + \underbrace{\vec{\omega}\times\vec{\omega}}_{=0} = \left(\dot{\vec{\omega}}\right)_{rotating}$$

Let us seek transformation of velocities

$$\vec{r}\,' = \vec{R} + \vec{r}$$

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{fixed} \quad \text{—} (14)$$

Now using (12)

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{r} \quad \text{—} (15)$$

Define

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} \equiv \vec{v}_f \equiv \mathring{\vec{r}}_f \quad \text{— (16a)}$$

$$\left(\frac{d\vec{R}}{dt}\right)_{fixed} \equiv \vec{V} = \mathring{\vec{R}}_f \quad \text{— (16b)}.$$

$$\left(\frac{d\vec{r}}{dt}\right)_{rotating} = \vec{v}_r = \mathring{\vec{r}}_r \quad \text{— (16c)}.$$

Can rewrite (15) as:

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}} \quad \text{— (17)}$$

$\vec{v}_f$: vel. w.r.t fixed axis

$\vec{V}$: Linear vel of moving origin

$\vec{v}_r$: vel. w.r.t rotating axis

$\vec{\omega} \times \vec{r}$: vel. due to rotation of moving axis

- $\vec{F} = m\vec{a}$ valid only in inertial reference frame
  in this case $\rightarrow$ fixed frame.

- $$\vec{F} = m\vec{a}_f = m\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} \qquad (18)$$

Recall eqn. (17)

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}$$

Differentiating, we get

$$\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} = \left(\frac{d\vec{V}}{dt}\right)_{fixed} + \left(\frac{d\vec{v}_r}{dt}\right)_{fixed} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} \qquad (19)$$

Recall eqn. 12

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q}$$

Define $\ddot{\vec{R}}_f = \left(\frac{d\vec{V}}{dt}\right)_{fixed}$ — 20

$$\left(\frac{d\vec{v}_r}{dt}\right)_{fixed} = \left(\frac{d\vec{v}_r}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}_r . \quad \text{— 21}$$

$$= \vec{a}_r + \vec{\omega} \times \vec{v}_r$$

$$\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{\omega} \times \vec{r} \quad \text{— 22}$$

Putting it all together (18) becomes.

$$\boxed{\vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + m\dot{\vec{\omega}}\times\vec{r} + m(\vec{\omega}\times(\vec{\omega}\times\vec{r})) + 2m\vec{\omega}\times\vec{v}_r}$$

— (23)

To an observer in the rotating coordinate system the "effective" force on the particle is

$$\vec{F}_{eff} = m\vec{a}_r. \quad — (24)$$

$$= \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_r.$$

— (25)

$-m\ddot{\vec{R}}_f \Rightarrow$ results from translational accln. of x_i system w.r.t x_i' system.

$-m(\dot{\vec{\omega}} \times \vec{r}) \Rightarrow$ results from rotational accln. of x_i system w.r.t x_i' system.

$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \Rightarrow$ centrifugal force term, familiar $m\omega^2 r$ $\vec{\omega} \perp \vec{r}$, -ve sign indicates direction outward.

$-2m\vec{\omega} \times \vec{v}_r \Rightarrow$ Coriolis force.

$\vec{F} = m\vec{a_f}$ valid in inertial frame

let

$\vec{F}_{eff} = m\vec{a_r}$ ($\ddot{\vec{R}_f}$ and $\dot{\vec{\omega}}$ be zero for simplicity).

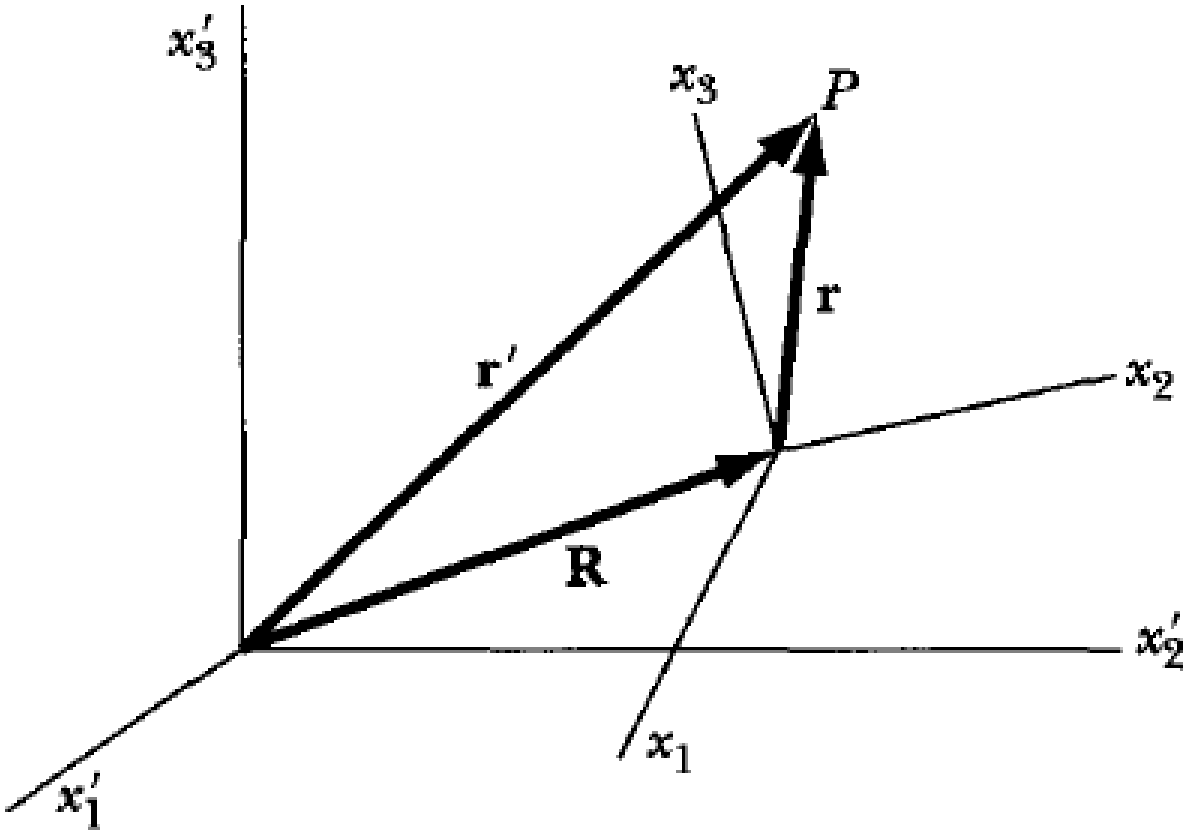
then

$\vec{F}_{eff} = m\vec{a_f}$ + (non-inertial terms)

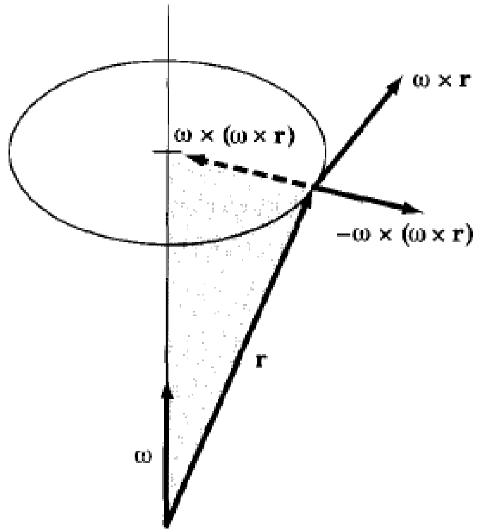
centrifugal + Coriolis

Physics I

Lecture 28



x_3'
x_3
P
$\mathbf{r}'$
$\mathbf{r}$
x_2
$\mathbf{R}$
x_2'
x_1
x_1'

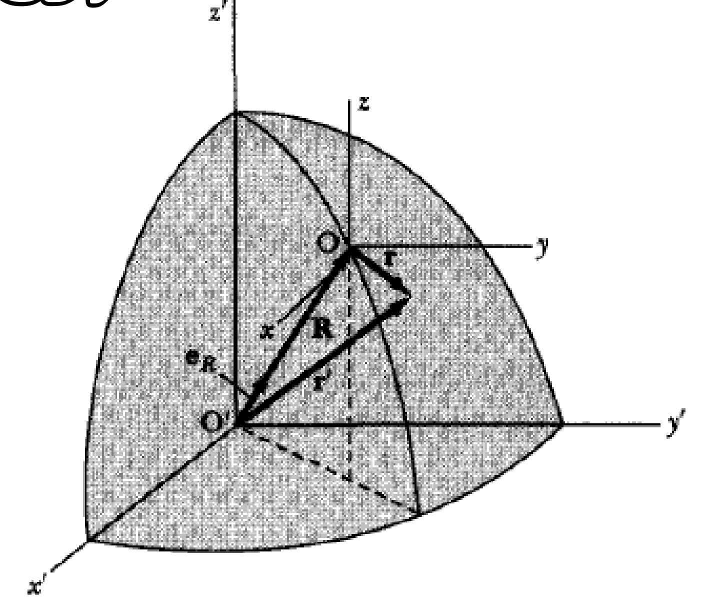


$$\vec{F}_{eff} = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \quad \text{—— (1)}$$

$$\vec{F}_{eff} = m\vec{a}_r \quad ; \quad \vec{F} = m\vec{a}_f \quad \text{—— (2)}$$

$$\vec{F}_{eff} = m\vec{a}_f + (\text{non-inertial terms}) \quad \text{— (3)}$$

Motion Relative to Earth



In order to study the motion of an object near Earth's surface, we place a fixed inertial frame $x'y'z'$ at the center of Earth and the moving frame xyz on Earth's surface.

$\vec{F} \longrightarrow$ forces measured w.r.t fixed inertial frame

$= \vec{S} + m\vec{g}_0$

$\hookrightarrow$ external forces other than gravitational

$$\vec{g}_0 = -\frac{GM_E}{R^2}\hat{e}_R \quad \text{—— (5)}$$

$$\vec{F}_{eff} = \vec{S} + m\vec{g}_0 - m\ddot{\vec{R}}_f - m\underbrace{\dot{\vec{\omega}} \times \vec{r}}_{\text{neglected } (\because \dot{\vec{\omega}} = 0)} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$- 2m\vec{\omega} \times \vec{v}_r \quad \text{—— (6)}$$

$\vec{\omega}$ is in z' direction ; $\omega = 7.3 \times 10^{-5}$ rad/s.

$\vec{\omega}$ is practically constant, $\dot{\vec{\omega}} = 0$.

Recall

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{Q} \quad \text{—— (7)}.$$

$$\therefore \ddot{\vec{R}}_f = \vec{\omega} \times \dot{\vec{R}}_f \quad \text{—— (8)}$$

$$\vec{F}_{eff} = \vec{S} + m\vec{g}_0 - m\vec{\omega} \times [\vec{\omega} \times (\vec{r} + \vec{R})] - 2m\vec{\omega} \times \vec{v}_r \quad \text{—— (9)}$$

$$\vec{F}_{eff} = \vec{S} + m\vec{g}_0 - m\vec{\omega} \times \left[\vec{\omega} \times (\vec{r} + \vec{R})\right] - 2m\vec{\omega} \times \vec{v}_r \quad (9)$$

effective $\vec{g}$ measured on earth.

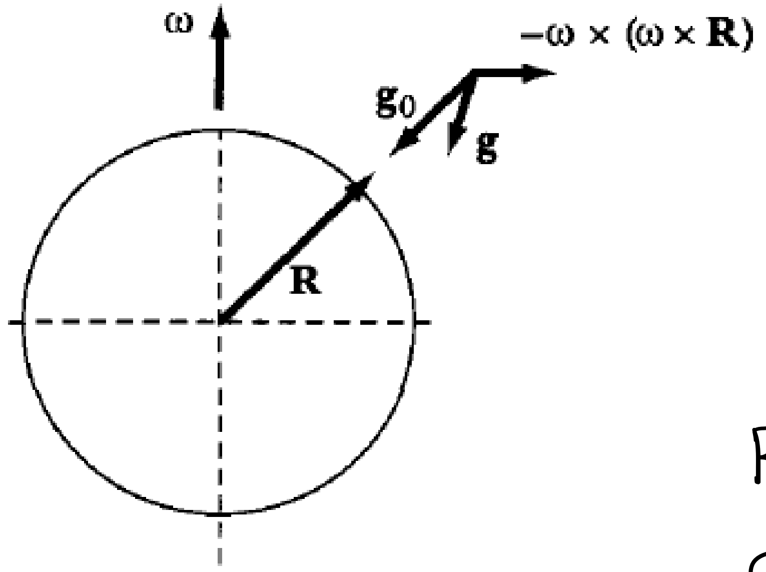
$$\vec{g} = \vec{g}_0 - \vec{\omega} \times \left[\vec{\omega} \times (\vec{r} + \vec{R})\right] \quad (10)$$

$$r \ll R \qquad \simeq \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

$$\boxed{\vec{F}_{eff} = \vec{S} + m\vec{g} - 2m\vec{\omega} \times \vec{v}_r} \quad (11)$$

Period of pendulum will determine mag. of g.

direction $\longrightarrow$ direction of a plumb bob.



$$\omega^2 R = 0.034\ \mathrm{m/s^2}$$

0.35% of g.

Relative magnitudes of centrifugal vs coriolis.

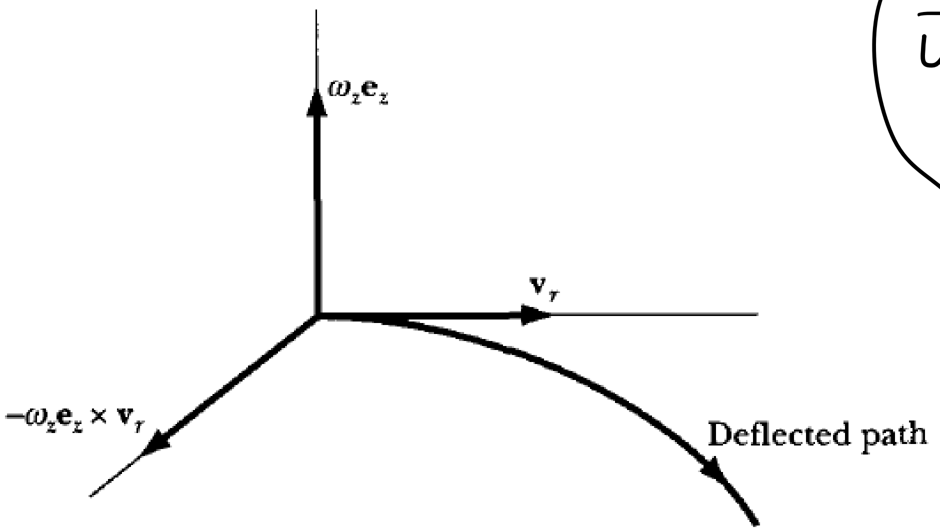
$$F_{cf} \sim m R \omega^2 .$$

$$F_c \sim m v \omega .$$

$$\frac{F_{cor.}}{F_c} \sim \frac{v}{R\omega} \sim \frac{v}{V} \sim \frac{v}{500\ \mathrm{m/s}} .$$

$v > 1800$ km/hr Coriolis force is imp.

Coriolis force effect



$\vec{\omega}$: directed in northerly direction

Northern hemisphere.

$\vec{\omega}$ has a component ω_z directed outward along local vertical.

If a particle is projected in a horizontal plane (in the local coord. system on surface of earth) $\vec{v}_r$

Coriolis force $= -2m\,\vec{\omega} \times \vec{v}_r$, has a component $2m\,\omega_z v_r$ directed towards the right

$\longrightarrow$ deflection to the right results.

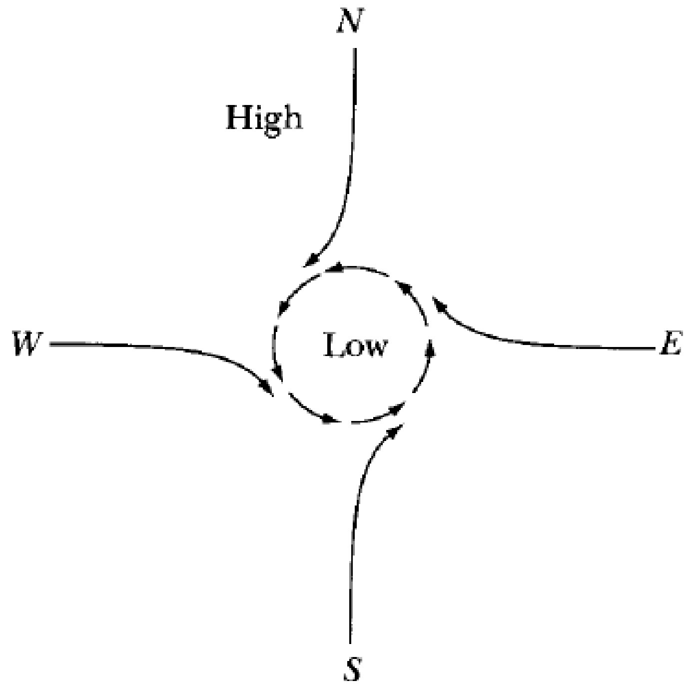
Coriolis force depends on z-component of ω.

$\Rightarrow$ depends on latitude, maximum at N-pole

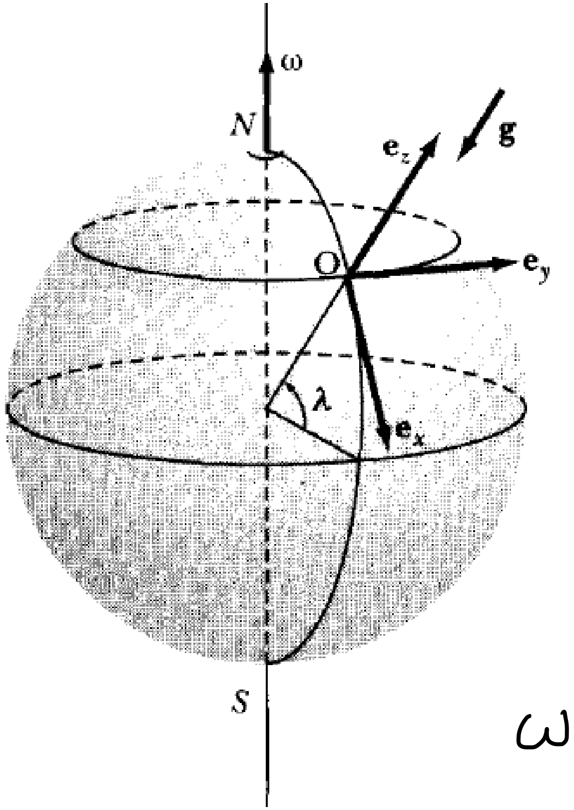
zero at equator.

In the Southern Hemisphere, the component of ω_z is directed inwards along the local vertical

$\Rightarrow$ all deflections will be to the left.
opposite to what happens in the N-hemisphere.



3 The Coriolis force deflects air in the Northern Hemisphere to the right producing cyclonic motion.



Horizontal deflection from the plumb line by the Coriolis force acting on a particle falling freely under earth's gravity.

$$\omega_x = -\omega\cos\lambda$$

$$\omega_y = 0$$

$$\omega_z = \omega\sin\lambda .$$

Eqn of motion

$$\ddot{x} = \underset{\approx 0}{g_x} - 2(\underline{\vec{\omega} \times \vec{v}_r})_x$$

$$\ddot{y} = \underset{=0}{g_y} - 2(\vec{\omega} \times \vec{v}_r)_y$$

$$\ddot{z} = g_z - 2(\vec{\omega} \times \vec{v}_r)_z$$

C-force produces small vel components in $\hat{e}_x$, $\hat{e}_y$ directions → ignore them

The zeroth approx we make

$$\left.\begin{aligned} \dot{x} &\simeq 0 \\ \dot{y} &\simeq 0 \\ \dot{z} &\simeq -gt \end{aligned}\right\}$$

$$\vec{\omega} \times \vec{v}_r \simeq \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\omega\cos\lambda & 0 & -\omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix}$$

$$\boxed{\vec{\omega} \times \vec{v}_r \approx -(\omega g t \cos\lambda)\hat{e}_y}$$

$g_x = 0$

$g_y = 0$

$g_z = -g.$

eqn. of motion

a

$$\left.\begin{aligned} (a_r)_x &= \ddot{x} \simeq 0 \\ (a_r)_y &= \ddot{y} \simeq 2\omega g t \cos\lambda \\ (a_r)_z &= \ddot{z} \simeq -g. \end{aligned}\right\}.$$

Integrate

$$\boxed{y(t) \simeq \frac{1}{3}\,\omega g t^3 \cos\lambda}$$

$[\, y = 0,\ \dot{y} = 0 \text{ at } t = 0 \,]$

$$z(t) \simeq \underbrace{z(0)}_{h} - \frac{1}{2} g t^2 .$$

time of fall $\quad t \simeq \sqrt{\dfrac{2h}{g}}.$

Eastward deflection

$$d \cong \frac{1}{3}\omega\, t^3 \cos\lambda \Big|_{t \cong \sqrt{\frac{2h}{g}}}$$

$$\boxed{d \cong \frac{1}{3}\omega \cos\lambda \sqrt{\frac{8h^3}{g}}}$$

$h \simeq 100\,m$ at latitude 45°

$$\boxed{d \simeq 1.55\ cm}$$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23
22
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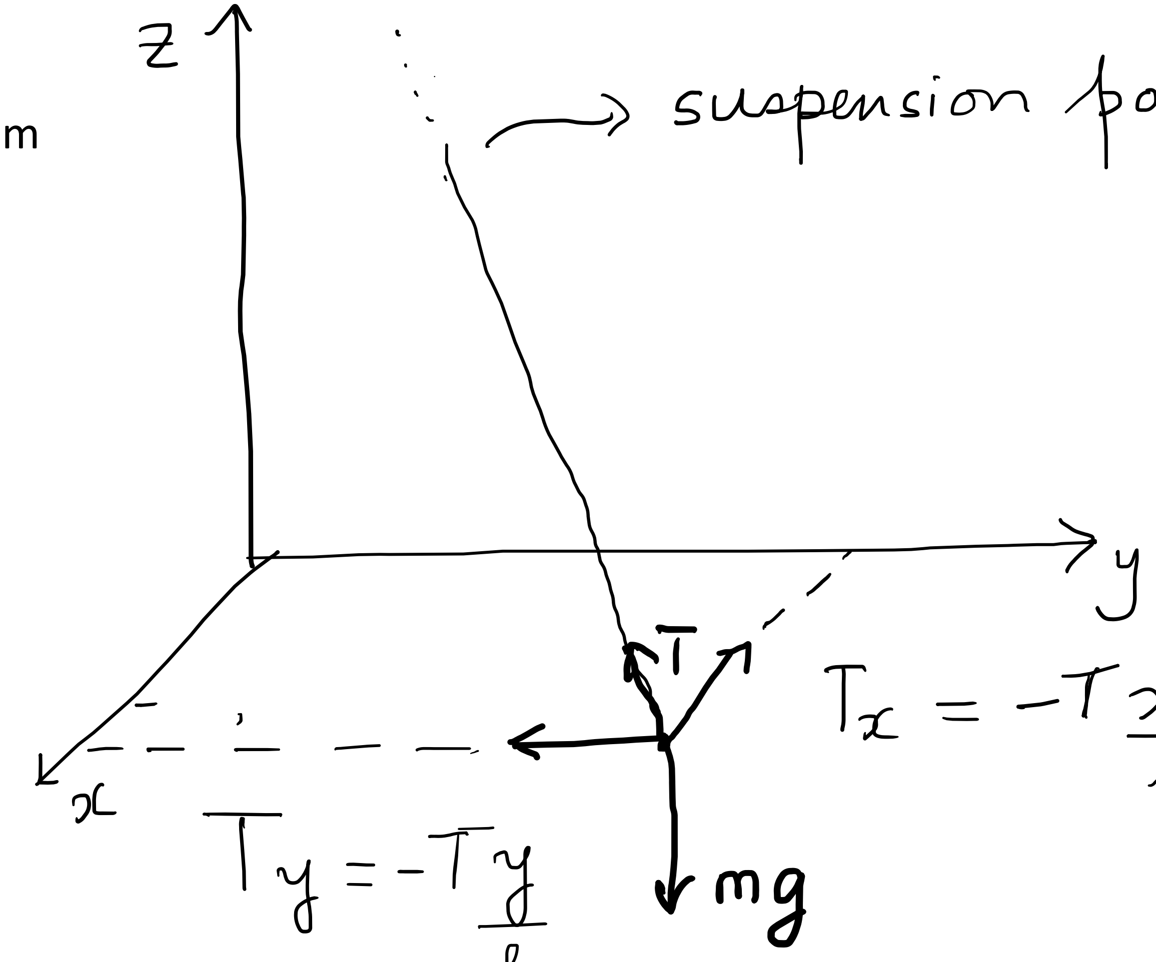
18
17

Physics I

Lecture 29

Foucault Pendulum



suspension point at great height

$\dot{z}$ small compared to $\dot{x}, \dot{y}$

$$T_x = -T\frac{x}{l}$$

$$T_y = -T\frac{y}{l}$$

$$\vec{a}_r = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_r .$$

$$T_x = -T\frac{x}{l}$$

$$T_y = -T\frac{y}{l}$$

$$T_z \cong T .$$

$$\vec{g}\ , \quad g_x = 0\ , \quad g_y = 0\ , \quad g_z = -g$$

$$\omega_x = -\omega\cos\lambda \longrightarrow \lambda : \text{latitude}.$$

$$\omega_y = 0.$$

$$\omega_z = \omega\sin\lambda.$$

$$(\vec{v}_r)_x = \dot{x}$$

$$(\vec{v}_r)_y = \dot{y}$$

$$(\vec{v}_r)_z = \dot{z} \cong 0$$

$$\vec{a}_r = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_r \quad \text{——} \quad \circledast$$

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

$$\left.\begin{aligned} (\vec{\omega} \times \vec{v}_r)_x &\cong -\dot{y}\,\omega\sin\lambda \\ (\vec{\omega} \times \vec{v}_r)_y &\cong \dot{x}\,\omega\sin\lambda \\ (\vec{\omega} \times \vec{v}_r)_z &\cong -\dot{y}\,\omega\cos\lambda \end{aligned}\right\}$$

$$\vec{a}_r = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_r .$$

$$\left.\begin{aligned} (\vec{a}_r)_x = \ddot{x} &\simeq -\frac{T}{m}\frac{x}{l} + 2\dot{y}\,\omega\sin\lambda \quad \text{— (1)} \\ (\vec{a}_r)_y = \ddot{y} &\simeq -\frac{T}{m}\frac{y}{l} - 2\dot{x}\,\omega\sin\lambda \quad \text{— (2)} \end{aligned}\right\}$$

for small displacements $T \simeq mg$.

$\alpha^2 = \dfrac{T}{ml} \simeq \dfrac{g}{l}$, $\omega_z = \omega\sin\lambda$; (1) & (2) become

$$\ddot{x} + \alpha^2 x \simeq 2\omega_z \dot{y} \quad \text{— (3)}$$

$$\ddot{y} + \alpha^2 y \simeq -2\omega_z \dot{x} \quad \text{— (4)}$$

multiply 2nd eqn. by i and add to 1st.

Becomes .

$$(\ddot{x} + i\ddot{y}) + \alpha^2 (x + iy) \simeq -2\omega_z (i\dot{x} - \dot{y})$$

$$\simeq -2i\omega_z (\dot{x} + i\dot{y}) \quad \text{—} \; ** $$

$$q \simeq x + iy$$

** becomes .

$$\ddot{q} + 2i\omega_z \dot{q} + \alpha^2 q = 0 \quad \text{—} \; 3$$

→ damped H.O. with pure imaginary damping coeff.

$$q(t) \cong e^{-i\omega_z t}\left[A e^{\sqrt{-\omega_z^2-\alpha^2}\,t} + B e^{-\sqrt{-\omega_z^2-\alpha^2}\,t}\right]$$

If the earth were not rotating $\omega_z = 0$.

$$\ddot{q}' + \alpha^2 q' \cong 0 \,,$$

$\alpha \Rightarrow$ oscillation freq. of pendulum $\gg \omega_z$.

$$q'(t) = x'(t) + i y'(t) = A e^{i\alpha t} + B e^{-i\alpha t}$$

$$q(t) = q'(t)\, e^{-i\omega_z t}.$$

$$x(t) + i y(t) = \left[x'(t) + i y'(t)\right] e^{-i\omega_z t}.$$

$$= \left[x'(t) + i y'(t)\right] \left(\cos\omega_z t - i \sin\omega_z t\right).$$

$$= \left(x' \cos\omega_z t + y' \sin\omega_z t\right)$$
$$+ \; i\left(-x' \sin\omega_z t + y' \cos\omega_z t\right)$$

$$\boxed{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\omega_z t & \sin\omega_z t \\ -\sin\omega_z t & \cos\omega_z t \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}}$$

$$\theta = \omega_z t = \omega \sin\lambda \, t.$$

Rigid Bodies

Example of many particle system is a rigid body.

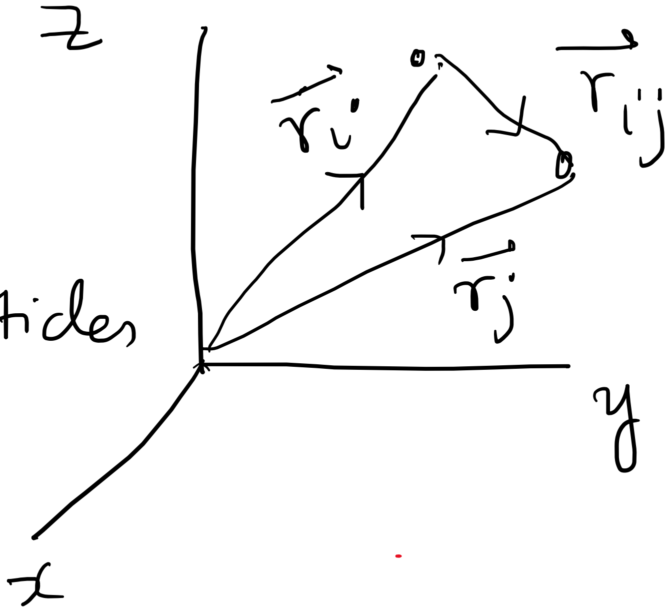
collection of particles whose relative distances are constrained to be fixed

Rigid body is an idealization

1. Component particles undergo vibrations.
2. In special relativity relative distances are observer dependent.

Number of degrees of freedom of a rigid body

N particles



If all particles were allowed to move freely

$\#$ of degrees of freedom $= 3N$

rigid body constraints

$|\vec{r}_{ij}| = c_{ij} = \text{constant}$ — (1)

$\#$ of constraints from (1).

$$= \frac{N(N-1)}{2}.$$

$\#$ of free degrees of freedom $= 3N - \frac{N(N-1)}{2}$?

for large N < 0

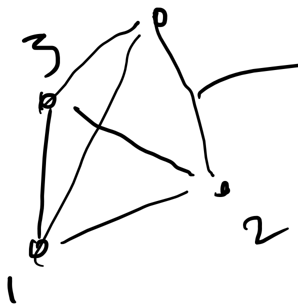
The constraints

$|\vec{r}_{ij}| = c_{ij}$ are not all independent

So what are the true # of degrees of freedom

$\boxed{2, 5 \quad ...??}$

One needs to fix coordinates of only 3 non colinear particles



$9 - 3 = 6$

Physics I

Lecture 30

Let us consider a rigid body composed of N particles of masses m_α, $\alpha = 1 \cdots N$.

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}} \quad \text{— (1)}$$

inst. vel of α^{th} particle in fixed system

rigid body rotates with an instantaneous ang vel $\vec{\omega}$ about some pt fixed w.r.t body coordinate system (origin), and this pt. moves with linear vel $\vec{V}$ w.r.t fixed inertial coordinate system.

But the rigid body condn.

$$\boxed{\vec{v_r} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} = 0} \rightarrow (2)$$

$\therefore$ from (1)

$$\boxed{\vec{V_\alpha} = \vec{V} + \vec{\omega} \times \vec{r_\alpha}} \quad (3)$$

K.E of α^{th} particle

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 \quad \text{--- (4)}$$

Total K.E (from (3))

$$T = \frac{1}{2} \sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \left(\vec{V} + \vec{\omega} \times \vec{r}_\alpha\right)^2 \quad \text{--- (5)}$$

$$= \frac{1}{2} \sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \vec{V} \cdot \left(\vec{\omega} \times \vec{r}_\alpha\right) + \frac{1}{2} \sum_\alpha m_\alpha \left(\vec{\omega} \times \vec{r}_\alpha\right)^2 \quad \text{--- (6)}$$

valid for arbitrary choice of origin

specializing to C.M as origin

$$2^{nd} \text{ term} = \vec{V} \cdot \vec{\omega} \times \underbrace{\sum_\alpha m_\alpha \vec{r}_\alpha}_{=0} \quad \text{--- (7)}$$

$$T = T_{trans} + T_{rot} \quad \text{— (8)}$$

$$\boxed{\begin{aligned} T_{trans} &= \frac{1}{2}\sum_{\alpha} m_\alpha V^2 = \frac{1}{2} M V^2 \\ T_{rot} &= \frac{1}{2}\sum_{\alpha} m_\alpha (\vec{\omega} \times \vec{r}_\alpha)^2 \end{aligned}} \quad \text{— (9)}$$

Using the identity $(\vec{A} \times \vec{B})^2 = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$.

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2\right] \quad \text{— (10)}$$

Express T_{rot} in components ω_i and $r_{\alpha i}$ of $\vec{\omega}$ and $\vec{r}_\alpha$: $\vec{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$

$$r_{\alpha,i} \equiv x_{\alpha,i} \text{ in body system.}$$

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right] \quad \text{— (11)}$$

can write $\omega_i = \sum_j \omega_j \delta_{ij}$, where $\delta_{ij} = 0 \; i \neq j$
$= 1 \; i = j$

$$T_{rot} = \frac{1}{2} \sum_{\alpha} \sum_{i,j} m_\alpha \left[\omega_i \omega_j \delta_{ij} \left(\sum_k x_{\alpha,k}^2 \right) - \omega_i \omega_j x_{\alpha,i} x_{\alpha,j} \right] \quad (12)$$

$$= \frac{1}{2} \sum_{i,j} \omega_i \omega_j \sum_{\alpha} m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \quad (13)$$

Define

$$\boxed{I_{ij} = \sum_{\alpha} m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)} \quad (14)$$

Now we have

$$\boxed{T_{tot} = \frac{1}{2} \sum_{i,j} I_{ij}\, \omega_i \omega_j} \quad (15)$$

Moment of inertia tensor.

$\{I\} \rightarrow$ matrix.

In restricted form

$$T_{rot} = \frac{1}{2} I \omega^2 \quad (16)$$

$$\{I\} \doteq \begin{bmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{bmatrix} \tag{17}$$

can be written in terms of

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha) .$$

$$\{I\} = \begin{Bmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum m_\alpha x_\alpha y_\alpha & -\sum m_\alpha x_\alpha z_\alpha \\ -\sum m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{Bmatrix} \qquad (18)$$

$\boxed{I_{ij} = I_{ji}}$

symmetric

Diagonal elements $\Rightarrow$ Moments of inertia (I_{11}, I_{22}, I_{33}).

-ve of Off diagonal elements $\Rightarrow$ products of inertia

Angular momentum

w.r.t some pt. O fixed in body coordinate system

$$\vec{L} = \sum_{\alpha} \vec{r}_{\alpha} \times \vec{p}_{\alpha} \quad \text{——} (19)$$

$$\vec{p}_{\alpha} = m_{\alpha} \vec{v}_{\alpha} = m_{\alpha} (\vec{\omega} \times \vec{r}_{\alpha}).$$

$$\vec{L} = \sum_{\alpha} m_{\alpha} \vec{r}_{\alpha} \times (\vec{\omega} \times \vec{r}_{\alpha}).$$

$$= \sum_{\alpha} m_{\alpha} \left[r_{\alpha}^2 \vec{\omega} - \vec{r}_{\alpha} (\vec{r}_{\alpha} \cdot \vec{\omega}) \right] \text{—} (20)$$

$$L_i = \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j}\, \omega_j \right)$$

$$= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j}\, \omega_j \right)$$

$$= \sum_j \omega_j \underbrace{\sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j} \right)}_{I_{ij}}$$

$$\boxed{L_i = \sum_j I_{ij}\, \omega_j} \quad \text{— } 21$$

$\left\{ \text{special case} \quad \vec{L} = I \vec{\omega} \right\}$

Physics I

Lecture 31

$$I_{ij} = \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{k} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)] \quad \text{—— } 1$$

$$T_{rot} = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \quad \text{—— } 2$$

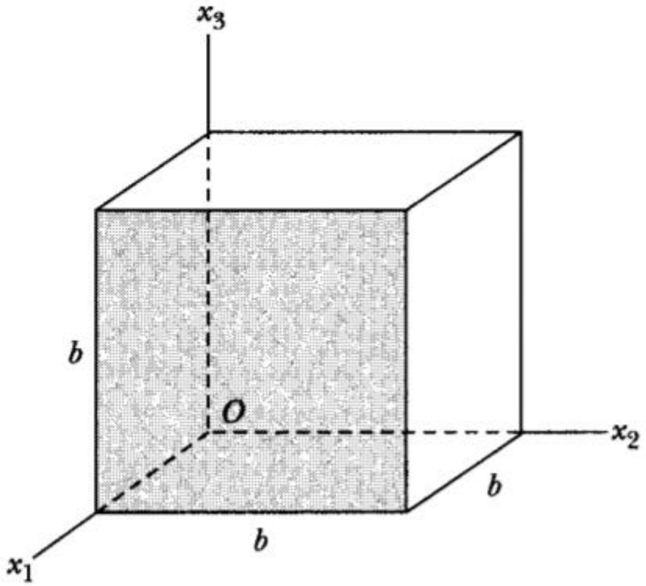
$$L_i = \sum_j I_{ij} \omega_j \quad \text{—— } 3$$

$$T = \frac{1}{2} \sum_i L_i \omega_i = \frac{1}{2} \vec{\omega} \cdot \vec{L} \quad \text{—— } 4$$

Continuum

$$I_{ij} = \int_V \rho(\vec{r}) \left(\delta_{ij} \sum x_k^2 - x_i x_j \right) dv \quad \text{—— } 5$$

$$dv = dx_1 dx_2 dx_3 .$$



homogeneous cube of density ρ, mass M, side b.

$$I_{ij} = \int_V dv\, \rho \left[\delta_{ij} \sum_k x_k^2 - x_i x_j \right]$$

$$I_{11} = \rho \int_0^b dx_1 \int_0^b dx_2 \int_0^b dx_3 \, (x_2^2 + x_3^2)$$

$$= \frac{2}{3} \rho\, b^5 \quad = \frac{2}{3} M b^2 \quad \text{—} \; (6)$$

Let

$\boxed{Mb^2 = \beta}$

$$I_{11} = I_{22} = I_{33} = \frac{2}{3} \beta \quad \text{—} \; (7)$$

All the off diagonal elements are equal too.

$$I_{12} = -\rho \int_0^b x_1 \, dx_1 \int_0^b x_2 \, dx_2 \int_0^b dx_3 .$$

$$= -\frac{1}{4} \rho b^5 = -\frac{1}{4} M b^2 . \quad \text{—— } 8$$

$$I_{12} = I_{13} = I_{23} = -\frac{1}{4}\beta \quad \text{—— } 9$$

$$\{I\} = \begin{Bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{Bmatrix} \quad 10$$

$$L_i = \sum_j I_{ij} \omega_j \quad \text{— (3)}$$

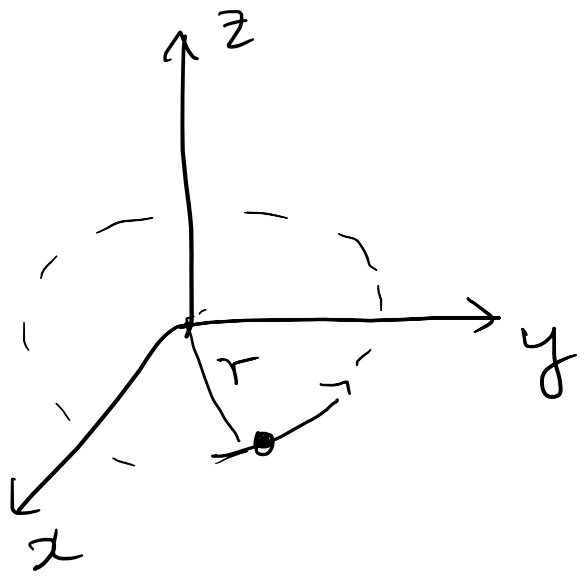
If the inertia tensor has non-vanishing off diagonal elements; then say $\vec{\omega} = (\omega_1, 0, 0)$.

$\vec{L}$ will have components in all directions.

$\{L_1, L_2, L_3\}$

> Angular momentum in general does not have same direction as ang. vel.

Example 1 (point mass in x-y plane)



↓ travelling in a circle of radius r, with freq. $\vec{\omega} = (0, 0, \omega)$.

$$x^2 + y^2 = r^2, \quad z = 0.$$

Using (3)

$$L_i = \sum_j I_{ij} \omega_j$$

$$L_x = I_{xz}\omega_z, \quad L_y = I_{yz}\omega_z, \quad L_z = I_{zz}\omega_z$$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j} \right).$$

$$I_{xz} = I_{yz} = 0 \qquad I_{zz} = mr^2$$

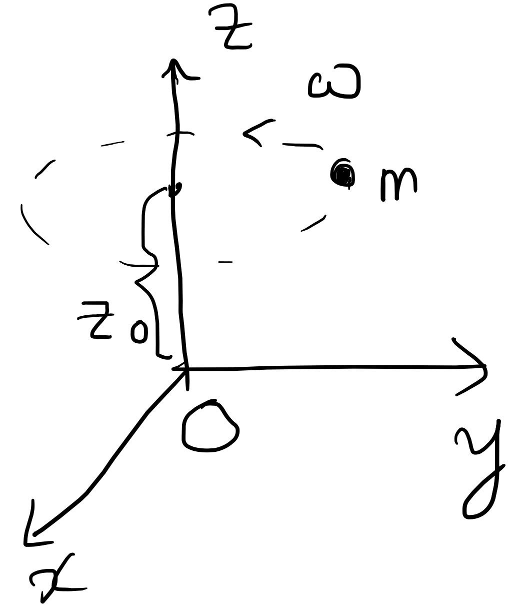
$$\boxed{L_z = mr^2\omega}$$

$L_x, L_y = 0$

$$\boxed{\vec{L} = mr^2\vec{\omega}}$$

$\vec{L}, \vec{\omega}$ in the same direction

Example 2. Point mass in space



$$\vec{\omega} = (0, 0, \vec{\omega}), \quad z = z_0, \quad r^2 = x^2 + y^2$$

$\vec{L}$ w.r.t O

$$I_{xz} = -m x z_0 \qquad I_{yz} = -m y z_0$$

$$I_{zz} = m r^2 .$$

$$\vec{L} = m\omega\left(-x z_0, -y z_0, r^2\right).$$

$$L_x, L_y \neq 0$$

$\vec{L}$ and $\vec{\omega}$ are not in the same direction.

Look through worked out examples 11.4 in Marion Thornton.

Principal Axes of Inertia

$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \quad \text{— (1)}$$

$$L_i = \sum_j I_{ij} \omega_j \quad \text{— (2)}$$

If I tensor had only diagonal elements

$$I_{ij} = I_i \delta_{ij} \Rightarrow (3) \qquad \{I\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix} \text{— (4)}$$

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 \quad \text{— (5)}$$

$$L_i = \sum_j \delta_{ij} I_i \omega_j = I_i \omega_i \quad \text{— (6)}$$

$\hookrightarrow$ find a set of body axes in which the products of inertia vanish

$\longrightarrow$ Principal axes of inertia.

$$\left.\begin{aligned} L_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 . \\ L_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\} \text{--- } 7$$

$\vec{L} = I\vec{\omega}$ — 8 body rotating about a principal axis

Combining 7, 8

$$\left.\begin{aligned} (I_{11}-I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0 \\ I_{21}\omega_1 + (I_{22}-I)\omega_2 + I_{23}\omega_3 = 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33}-I)\omega_3 = 0 \end{aligned}\right\} \text{ 9}$$

Eqns (9) will have non trivial solns provided

$$\begin{vmatrix} (I_{11}-I) & I_{12} & I_{13} \\ I_{21} & (I_{22}-I) & I_{23} \\ I_{31} & I_{32} & (I_{33}-I) \end{vmatrix} = 0 \quad \text{— (1)}$$

→ secular eqn is cubic, each root is called a principal moment of inertia (I_1, I_2, I_3).

→ directions are determined by eigenvectors

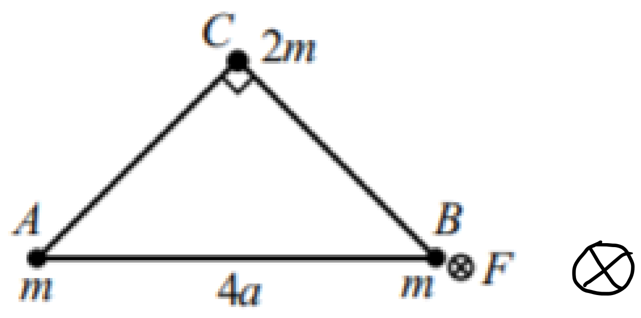
Physics I

Lecture 32

Two classes of problems

1. Strike a rigid object with an impulsive blow and ask what is the motion of the object immediately after the blow.

2. An object rotates about a fixed ~~axsis~~ axis. A given torque is applied to it. What is the frequency of rotation.

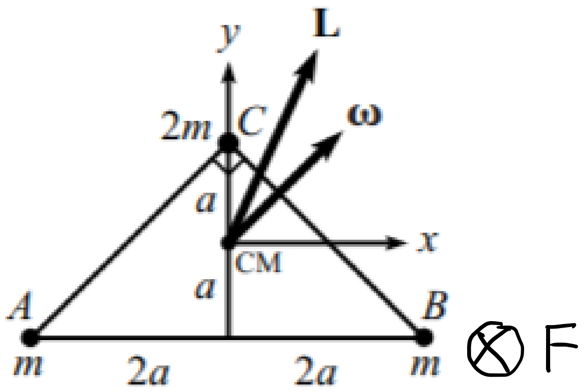


- Three massless rods arranged in an isosceles right angled triangle, with given masses at vertices.
- floating freely in space.
- Impulse mag $\int F\,dt = P$

What are velocities of the 3 masses after the blow.

Strategy

- Find angular momentum relative to CM.
- Identify principal axes and calculate principal moments of inertia
- find $\vec{\omega}$ from $\vec{L} = \{I\}\vec{\omega}$
- find $\vec{v}$ from $\vec{\omega}$.
- Then add on CM motion.



x-y plane, z axis out of the plane.

- CM lies at mid pt of altitude.
- CM as origin.

Positions of masses.

$\vec{r}_A = (-2a, -a, 0)$, $\vec{r}_B = (2a, -a, 0)$

$\vec{r}_C = (0, a, 0)$.

1. Find $\vec{L}$; $\dfrac{d\vec{L}}{dt} = \vec{\tau}$, $\vec{L} = \int \vec{\tau}\, dt$.

$\vec{L} = \int \vec{\tau}\, dt = \int \vec{r}_B \times \vec{F}\, dt$; $\vec{P} = \int \vec{F}\, dt = (0, 0, -P)$

$= \vec{r}_B \times \underbrace{\int \vec{F}\, dt}_{P} \rightarrow$ [sudden implulse $\vec{r}_B$ stays const]

$$\boxed{\vec{L} = (2a, -a, 0) \times (0, 0 - P) = aP(1, 2, 0)}$$

2. Find Principal moments.

↳ Principal axes are x, y, z. Symmetry makes I diagonal in this basis.

$$I_x = ma^2 + ma^2 + 2ma^2 = 4ma^2$$

$$I_y = m(2a)^2 + m(2a)^2 + 2m \cdot 0 = 8ma^2.$$

$$I_z = I_x + I_y = 12ma^2.$$

3. Find $\vec{\omega}$ from $\vec{L} = \{I\}\vec{\omega}$ $\vec{L} = (aP, 2aP, 0)$

$$\vec{L} = aP\,\hat{x} + 2aP\,\hat{y} + 0\,\hat{z} = I_x\omega_x\,\hat{x} + I_y\omega_y\,\hat{y} + I_z\omega_z\,\hat{z}$$

$$= 4ma^2\omega_x\,\hat{x} + 8ma^2\omega_y\,\hat{y} + 12ma^2\omega_z\,\hat{z}$$

$$\omega_x = \frac{aP}{4ma^2}, \quad \omega_y = \frac{2aP}{8ma^2}, \quad \omega_z = 0.$$

$$(\omega_x, \omega_y, \omega_z) = \frac{P}{4ma}(1, 1, 0).$$

4. Calculate velocities w.r.t CM.

Right after blow, object rotates about the CM with ang vel found in step 3.

$$\vec{u}_i = \vec{\omega} \times \vec{r}_i$$

$$\vec{u}_A = \vec{\omega} \times \vec{r}_A = \frac{P}{4ma}(1,1,0) \times (-2a,-a,0) = \left(0,0,\frac{P}{4m}\right)$$

$$\vec{u}_B = \vec{\omega} \times \vec{r}_B = \frac{P}{4ma}(1,1,0) \times (2a,-a,0) = \left(0,0,\frac{-3P}{4m}\right)$$

$$\vec{u}_C = \vec{\omega} \times \vec{r}_C = \frac{P}{4ma}(1,1,0) \times (a,a,0) = \left(0,0,\frac{P}{4m}\right)$$

5. Add on vel of CM.

$\vec{P} = (0, 0, -P)$. Total mass $= M = 4m$

Impulse = change in momentum.

$\vec{P} = M\vec{V}_{cm}$, $\vec{V}_{cm} = \dfrac{\vec{P}}{M} = \left(0, 0, -\dfrac{P}{4m}\right)$

Total vel. of masses.

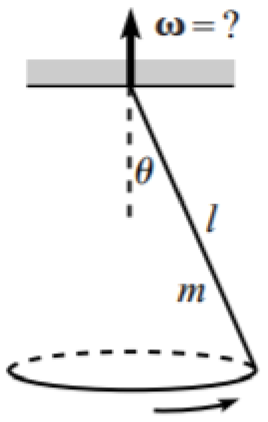
$\vec{v}_A = \vec{u}_A + \vec{V}_{CM}. = (0, 0, 0)$.

$\vec{v}_B = \vec{u}_B + \vec{V}_{CM} = \left(0, 0, \dfrac{-P}{m}\right)$

$\vec{v}_C = \vec{u}_C + \vec{V}_{CM} = (0, 0, 0)$

Stick of length l, mass m of uniform density.
pivoted at its top end swings around a vertical axis. such that the stick always makes a constant angle θ with vertical.

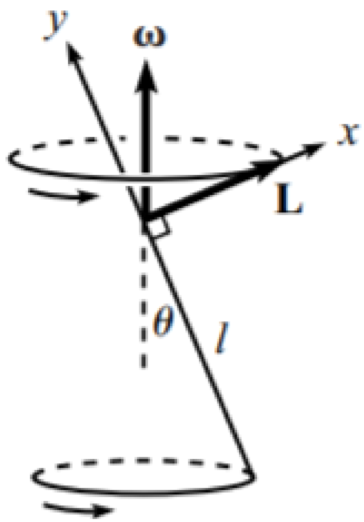
$$\vec{\omega} = ?$$



$$\vec{\omega} = ?$$

Strategy

1. Find Principal moments
2. Find $\vec{L}$
3. Find $\frac{d\vec{L}}{dt}$.
4. Use $\vec{\tau} = \frac{d\vec{L}}{dt}$.



- Origin as pivot point.
- Axis along stick + any two axes orthogonal to it as principal axes.
- +z points out of the page.

1. Find Principal moments.

$$I_x = \frac{ml^2}{3}, \quad I_y = 0, \quad I_z = \frac{ml^2}{3}.$$

2. Find $\vec{L}$, $\quad \vec{L} = \{I\}\vec{\omega}$.

$\vec{\omega} = (\omega \sin\theta, \omega\cos\theta, 0)$, $\quad L_x = I_x \omega_x, \; L_y = I_y \omega_y$
$L_z = I_z \omega_z$.

$$\vec{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z)$$

$$= \left(\tfrac{1}{3} ml^2\omega\sin\theta, 0, 0\right).$$

Next $\frac{d\vec{L}}{dt}$.

as stick rotates, $\vec{L}$ traces out surface of cone.

tip traces out circle of radius $L\cos\theta$.

speed of tip $= (L\cos\theta)\omega$.

$$\left|\frac{d\vec{L}}{dt}\right| = (L\cos\theta)\omega = \tfrac{1}{3} ml^2\omega^2\sin\theta\cos\theta.$$

points into the page.

Or. alternatively

$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L}$$

$$= (\omega \sin\theta, \omega\cos\theta, 0) \times \left(\frac{1}{3} m l^2 \omega \sin\theta, 0, 0\right)$$

$$= \left(0, 0, -\frac{1}{3} m l^2 \omega^2 \sin\theta \cos\theta\right)$$

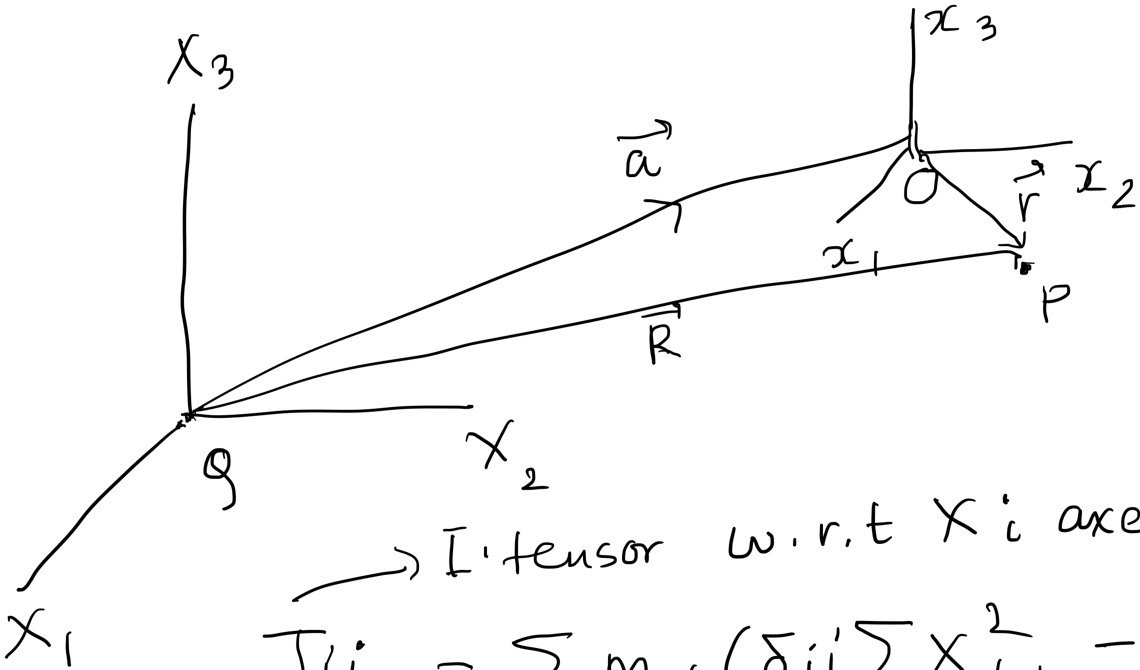
Next, calculate torque relative to pivot.

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF\sin\theta = \frac{l}{2} m g \sin\theta$$; points into page.

$$\vec{\tau} = \frac{d\vec{L}}{dt} ; \quad \cancel{m} \frac{l^{\cancel{2}} \omega^2 \cancel{\sin\theta} \cos\theta}{3} = \frac{\cancel{m} g \cancel{l} \cancel{\sin\theta}}{2}$$

$$\boxed{\omega = \sqrt{\frac{3g}{2l\cos\theta}}}$$

Generalized Parallel-Axis Theorem



I tensor w.r.t X_i axes.

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j}\right) \quad \text{—(1)}$$

$$\vec{R} = \vec{a} + \vec{r} \quad \text{—(2)} \qquad \vec{R} = (X_1, X_2, X_3)$$

$$\vec{r} = (x_1, x_2, x_3)$$

$$X_i = a_i + x_i \quad \text{—(3)}$$

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k (x_{\alpha,k} + a_k)^2 - (x_{\alpha,i} + a_i)(x_{\alpha,j} + a_j)\right)$$

$$= \underbrace{\sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right)}_{I_{ij}}$$

$$+ \sum_\alpha m_\alpha \Big\{ \Big(\delta_{ij} \sum_k (2 x_{\alpha,k} a_k + a_k^2)\Big) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j) \Big\} \quad \text{—(4)}$$

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j\right)$$

$$+ \sum_\alpha m_\alpha \left(2 \delta_{ij} \sum_k x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i}\right)$$

In the last summation each teen involves $\sum_\alpha m_\alpha x_{\alpha,k} = 0$, $\sum m_\alpha \vec{r}_\alpha = 0$

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j\right)$$

$$\sum m_\alpha = M, \quad \sum_k a_k^2 = a^2.$$

$$\boxed{I_{ij} = J_{ij} - M\left(a^2 \delta_{ij} - a_i a_j\right)}$$

<u>Th. 1</u> If two principal moments are equal ($I_1 = I_2 = I$) then any axis (through the chosen origin) in the plane of the corresponding principal axis, is <u>also</u> a principal axis, and its moment is also I.

Proof: $\because \quad I_1 = I_2 = I$

if $\vec{u_1}$ and $\vec{u_2}$ are eigenvectors of $\{I\}$

$$\{I\}\vec{u_1} = I\vec{u_1} \quad ; \quad \{I\}\vec{u_2} = I\vec{u_2}$$

$$\{I\}\left(a\vec{u_1} + b\vec{u_2}\right) = I\left(a\vec{u_1} + b\vec{u_2}\right)$$

for all a, b.

$\rightarrow$ any vector in plane spanned by $\vec{u_1}$ & $\vec{u_2}$ is also a soln. $\Longrightarrow$ principal axis

Th.2. If a pancake object is symmetric under a rotation through $\theta \neq 180°$ in the x-y plane, then every axis in the x-y plane (with origin at the centre of symetry rotation) is a principal axis with same moment.

$\vec{\omega_0}$: principal axis in plane
$\vec{\omega_\theta}$; axis obtained by rotating $\vec{\omega_0}$ through θ

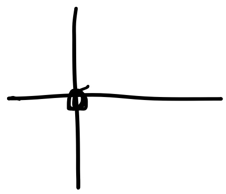
$$\therefore \quad \{I\}\vec{\omega_0} = I\vec{\omega_0}$$
$$\{I\}\vec{\omega_\theta} = I\vec{\omega_\theta}$$

Any vector $\vec{\omega}$ in x-y plane can be written as a linear combination of $\vec{\omega_0}$ and $\vec{\omega_\theta}$, provided that $\theta \neq 180°$ or 0. $\vec{\omega_0}, \vec{\omega_\theta}$ span the plane.

$$I\vec{\omega} = \{I\}(a\vec{\omega_0} + b\vec{\omega_\theta}) = aI\vec{\omega_0} + bI\vec{\omega_\theta}$$

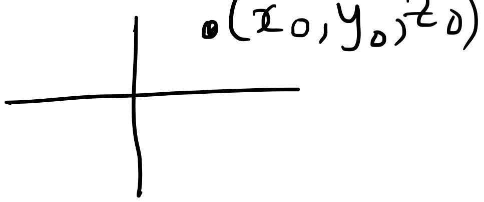
Hence $\vec{\omega}$ is also a principal axis.

Ex :



point mass at origin

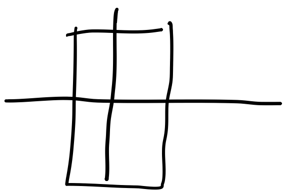
Any axis principal axis



pt mass at (x_0, y_0, z_0).

Axis through the pt.

any axis $\perp$ to it



Rectangle centred at origin

P.A : x, y, z axis

Diagonalized $\{I\}$

$$I_{ij} = I_i \delta_{ij} \quad \text{— (1)}$$

(I_1, I_2, I_3) principal moments of inertia

direction of each principal axis is determined by substituting I_1, I_2, I_3 for I in the eqn.

$$I\omega_1 = I_{11}\omega_1 , \; I\omega_2 = I_{22}\omega_2 , \; I\omega_3 = I_3\omega_3$$

$\hookrightarrow$ determines ratios of ang. vel. vector

Principal axes $\Longrightarrow$ eigenvectors.

$I_1 = I_2 = I_3 \Rightarrow$ spherical top.

$I_1 = I_2 \neq I_3 \Rightarrow$ symmetric top.

$I_1 \neq I_2 \neq I_3 \Rightarrow$ asymmetric top.

<u>Generalized Parallel-axis Theorem</u>