Physics I

Lecture 10

Harmonic oscillator in 1D.

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_0^2 x = 0$$

$$\to \quad x = A\cos\omega_0 t + B\sin\omega_0 t$$

$$\text{or} \quad \tilde{A}\sin(\omega_0 t - \delta).$$

Harmonic oscillator in 2D

$$\vec{F} = -k\vec{r} \qquad F_x = -kx \;, \; F_y = -ky$$

$$\left.\begin{aligned} x(t) &= A\cos(\omega_0 t - \alpha) \\ y(t) &= B\cos(\omega_0 t - \beta) \end{aligned}\right\}$$

Please read this section in book.

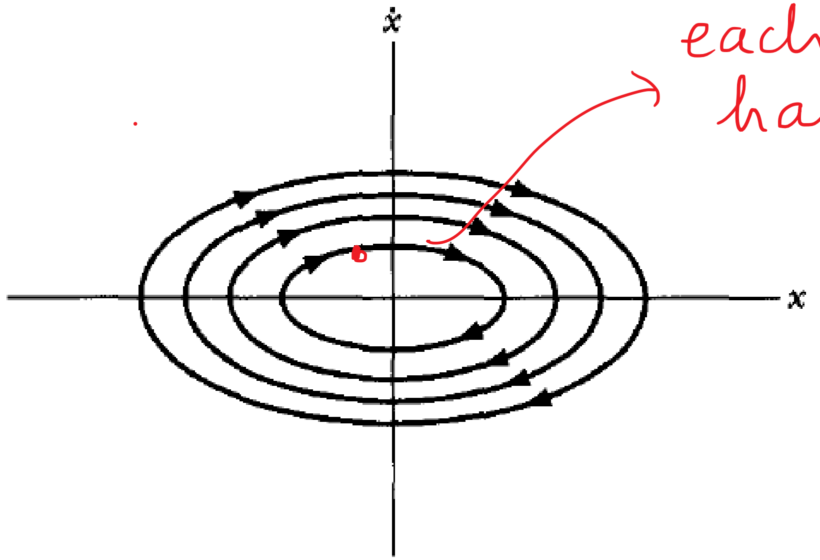
Phase diagrams

$$F = -kx$$

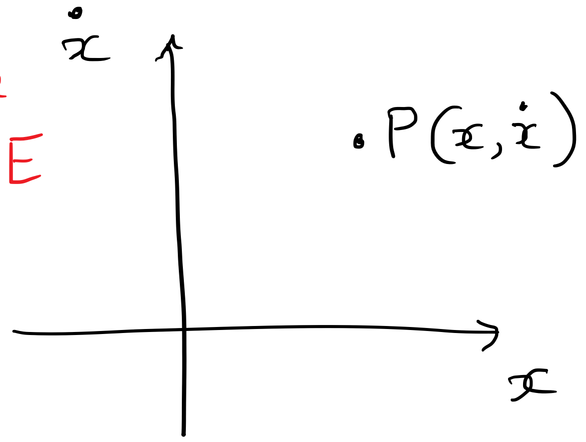
$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\boxed{E = \frac{1}{2}kA^2}$$

const.



each ellipse has fixed E



For any dynamical system, specifying $x, \dot{x}$

$x(t_0), \dot{x}(t_0)$ complete specification of state

$(x, \dot{x}) \longrightarrow$ phase space, In the case of 1-d motion phase space is 2D.

[gas: N particles in 3 D

dimension of phase space $\equiv 6N$]

$$x = A\sin(\omega_0 t - \delta)$$

$$\dot{x} = A\omega_0 \cos(\omega_0 t - \delta).$$

eliminate t

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1 \longrightarrow \text{ellipse in phase space.}$$

Recall $E \propto A^2$

Two phase trajectories can never cross.

clockwise trajectories, $x > 0$, $\dot{x}$ decreasing

$x < 0$ $\dot{x}$ increasing

Damped oscillations

Undamped case

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_0^2 x = 0$$

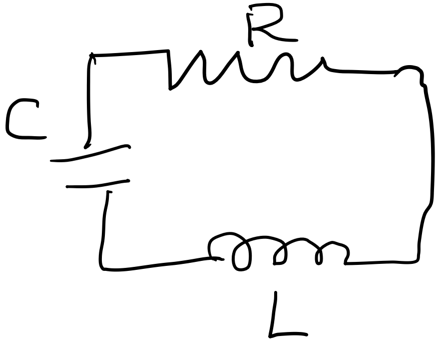
Add damping

$$m\ddot{x} = -kx - b\dot{x}$$

$$\boxed{m\ddot{x} + b\dot{x} + kx = 0}$$

Interesting analogy

LCR circuit



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$$

$q \to x$

$L \to m$

$R \to b$

$\frac{1}{C} \to k$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$2\beta = \frac{b}{m}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 = \frac{k}{m}.$$

Let $x = e^{pt}$

$$p^2 + 2\beta p + \omega_0^2 = 0 \implies \text{auxilliary eqn.}$$

two solns

$$\left.\begin{aligned} p_1 &= -\beta + \sqrt{\beta^2 - \omega_0^2} \\ p_2 &= -\beta - \sqrt{\beta^2 - \omega_0^2} \end{aligned}\right\} \quad e^{p_1 t},\ e^{p_2 t} \text{ solns.}$$

General soln:

$$x(t) = C_1 e^{P_1 t} + C_2 e^{P_2 t}.$$

$$\boxed{x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)}$$

Cases :

1. undamped, $\beta = 0$, $x = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}$
2. underdamped $\beta^2 < \omega_0^2$
3. critically damped $\beta = \omega_0$
4. Overdamped $\beta^2 > \omega_0^2$.

Underdamped

Define $\sqrt{\beta^2-\omega_0^2} = i\sqrt{\omega_0^2-\beta^2} = i\omega_1$

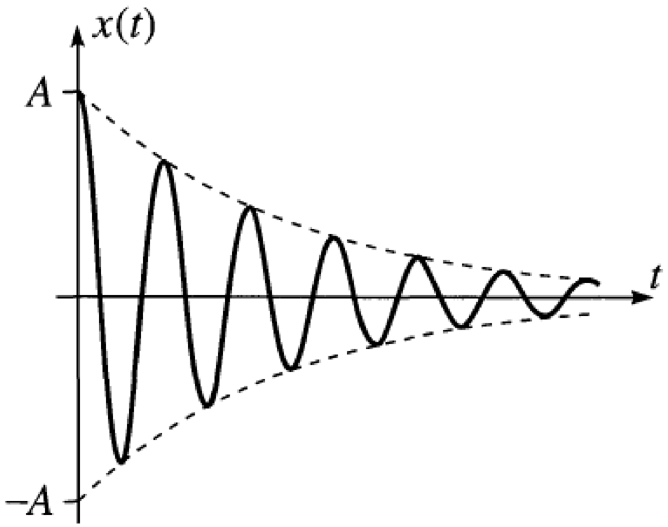
$$x(t) = e^{-\beta t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$$

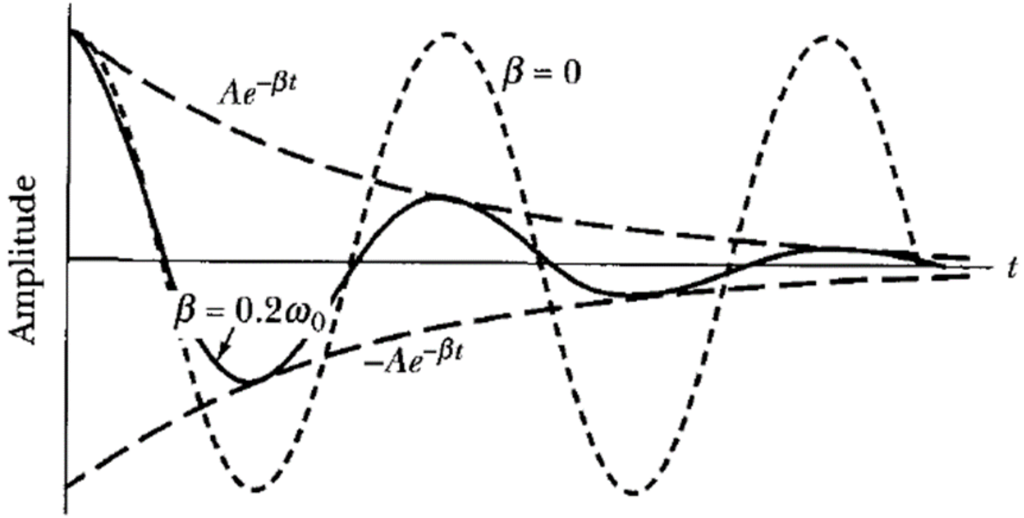
writing $C_1 = \frac{A}{2}e^{-i\delta}$, $C_2 = \frac{A}{2}e^{+i\delta}$

$$\boxed{x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)}$$

decaying amplitude.

β has dimensions of t^{-1}, $\frac{1}{\beta}$ is the time in which the amplitude falls to $\frac{1}{e}A$





underdamped case.

$$x_{env} = \pm A\, e^{-\beta t}.$$

notice that $\omega_1 < \omega_0$.

Undamped $E = \frac{1}{2}kA^2 \longrightarrow$ conserved.

$\beta \ll \omega_0$ $\quad \omega_1 \simeq \omega_0$

$$E \simeq \frac{1}{2}kA^2 e^{-2\beta t}$$

Critically damped

$\beta = \omega_0$ coincident roots. one soln. $x = e^{-\beta t}$

Another linearly ind. soln $x = t e^{-\beta t}$

$x(t) = c_1 e^{-\beta t} + c_2 t e^{-\beta t}$

$$\boxed{x(t) = e^{-\beta t}\,(c_1 + c_2 t)}$$

decay parameter $= \beta$

Overdamped case

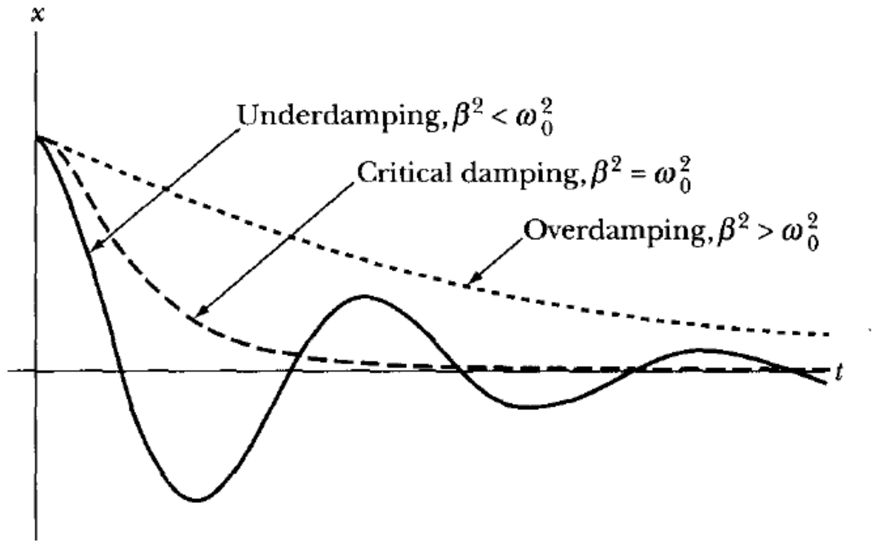
$\beta > \omega_0$ sq. root is real

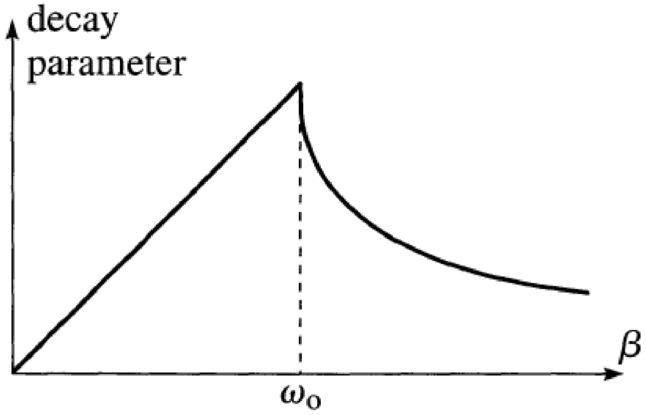
$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}.$$

exponential decay.

first term decays slower, dominates.

decay parameter $\beta - \sqrt{\beta^2 - \omega_0^2}$.





decay parameter

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$