Physics I

Lecture 11

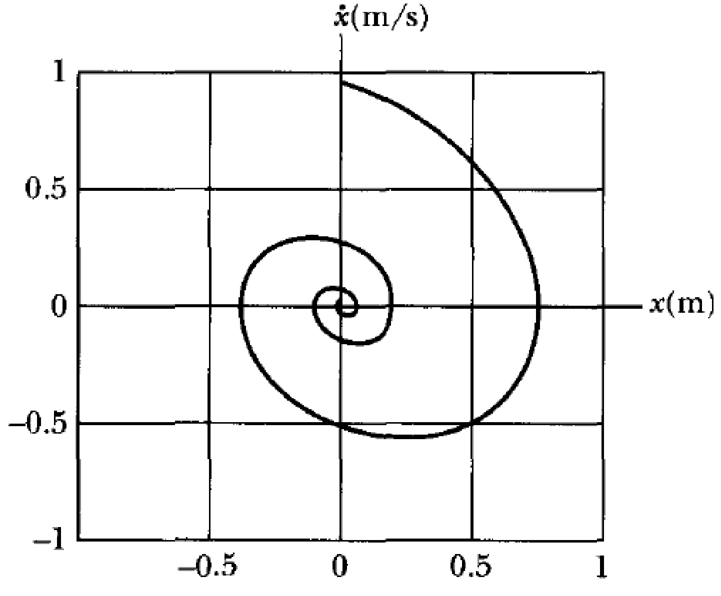
Damped harmonic oscillator (recap)

$$\ddot{x} + 2\beta x + \omega_0^2 x = 0$$

- Underdamped $\beta < \omega_0$
- Critically damped $\beta = \omega_0$
- overdamped $\beta > \omega_0$

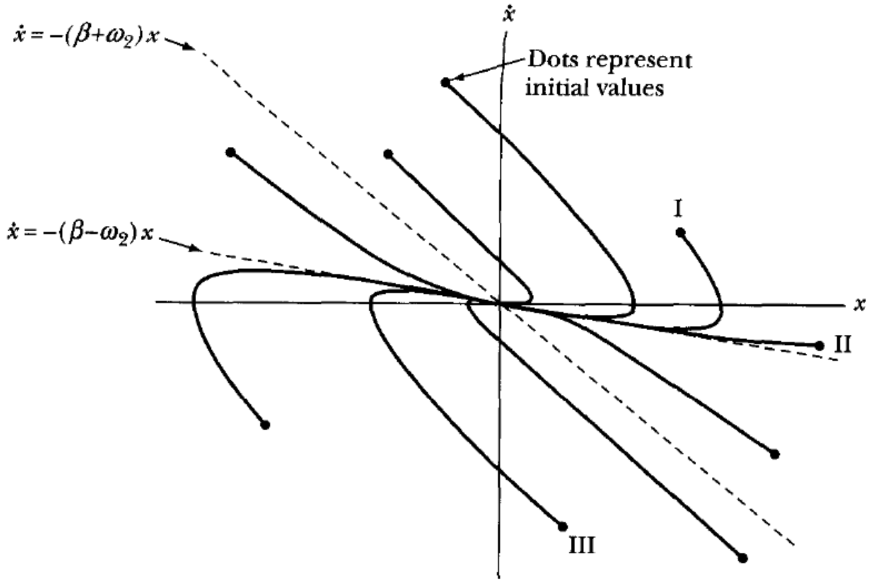
PHASE SPACE PLOTS

$\omega_2 = \sqrt{\beta^2 - \omega_0^2}$



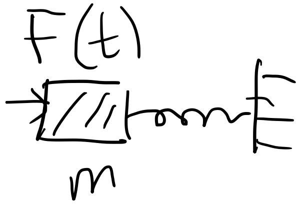
(b)

underdamped



overdamped

Forced/Driven Damped harmonic oscillator



$$m\ddot{x} + b\dot{x} + kx = F(t)$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \qquad f(t) = \frac{F}{m}.$$

$\hookrightarrow$ inhomogeneous diff eqn.

<u>Theorem</u> : If $x_p(t)$ (particular soln.) is a solution of an inhomogeneous diff. eqn. and $x_h(t)$ is a soln. to the corresponding homogeneous eqn., then $x_p(t) + x_h(t)$ is also a soln. to the inhomogeneous eqn..

$$\boxed{\text{General soln.}: \quad x_h(t) + x_p(t)}$$

We will specialize to

$$f(t) = f_0 \cos\omega t$$

ω ; driving frequency.

$$\boxed{\frac{F(t)}{m} = f(t)}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

$\operatorname{Re}(f_0 e^{i\omega t})$.

Assume a complex soln. of the form $z = Ce^{i\omega t}$ — (1)

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t} \quad \text{— (2)}$$

Plug (1) into (2)

$$(-\omega^2 + 2i\beta\omega + \omega_0^2) Ce^{i\omega t} = f_0 e^{i\omega t}$$

$z = C e^{i\omega t}$ is a soln, provided

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = A e^{-i\delta}$$ A, δ real.

$$\boxed{A^2 = CC^* = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad (3)$$

$f_0 e^{i\delta} = A(\omega_0^2 - \omega^2 + 2i\beta\omega)$

check that

$$\boxed{\delta = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}} \quad (4)$$

Soln:

$$\boxed{z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}} \quad (4)$$

Homogeneous soln:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$x \sim e^{pt}$$

$$x_h = C_1 e^{p_1 t} + C_2 e^{p_2 t} \quad (5)$$

$$\boxed{p_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}}$$

Full soln: $x_p(t) + x_h(t)$

dies out at late time

transient

Let us specialize to underdamped case

$$\boxed{x(t) = A\cos(\omega t - \delta) + A_{tr}e^{-\beta t}\cos(\omega_1 t - \delta_{tr})} \quad (6)$$

Arbitrary const to be fixed by initial conditions

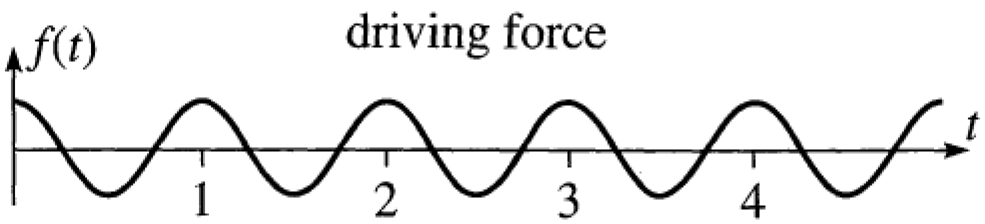
{ Recall underdamped soln:

$$x(t) = Ce^{-\beta t}\cos(\omega_1 t - \phi) \}$$

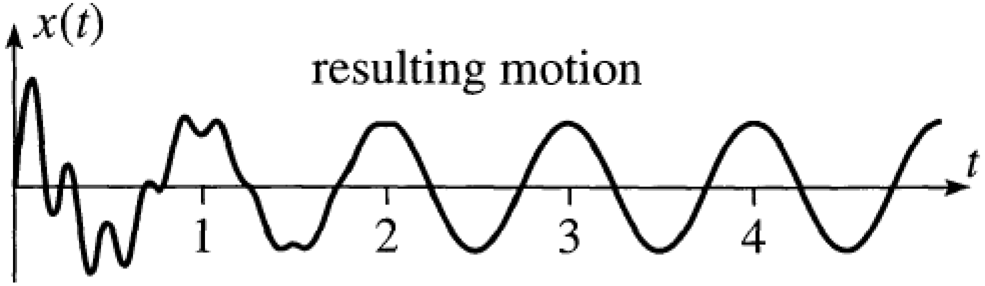
$C \equiv A_{tr}$ tr: transient

$\phi \equiv \delta_{tr}$ }

wipes out memory of initial conditions.



(a)



(b)

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

- $A \propto f_0$
- phase lag between the driving force and resulting motion

$\rightarrow f_0 \cos(\omega t)$

$\rightarrow A\cos(\omega t - \delta)$

Resonance

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Maximum Amplitude

$$\left.\frac{dA}{d\omega}\right|_{\omega=\omega_R} = 0 \quad \rightsquigarrow \quad \boxed{\omega_R = \sqrt{\omega_0^2 - 2\beta^2}}$$

Res. freq is lowered as β increases

No res. will occur for $\beta > \frac{\omega_0}{2}$.

1. free oscillations, no damping

$$\omega_0^2 = \frac{k}{m}$$

2. + damping

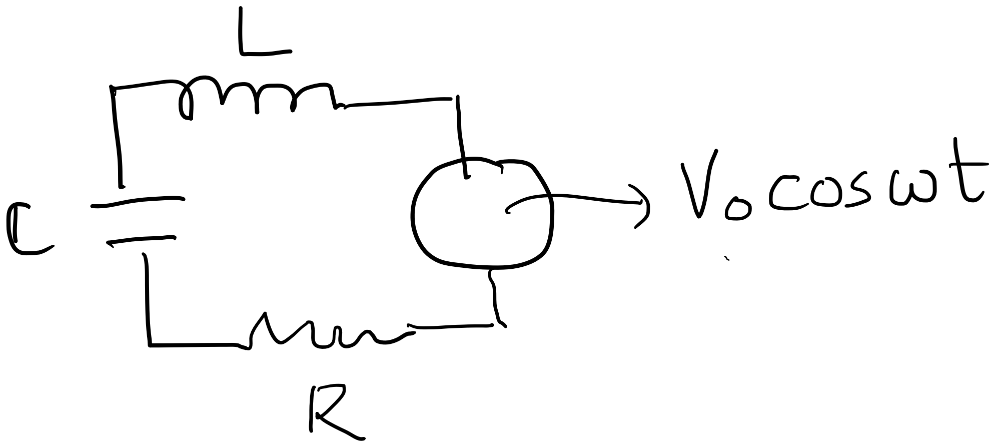
$$\omega_1^2 = \omega_0^2 - \beta^2$$

3. + Driving force.

$$\omega_R^2 = \omega_0^2 - 2\beta^2$$

$$\omega_0 > \omega_1 > \omega_R.$$

Analog LCR



$$m \equiv L$$

$$k \equiv \frac{1}{C}$$

$$R = 2\beta \,.$$

$$\frac{V_0}{L} \equiv f_0 \,.$$

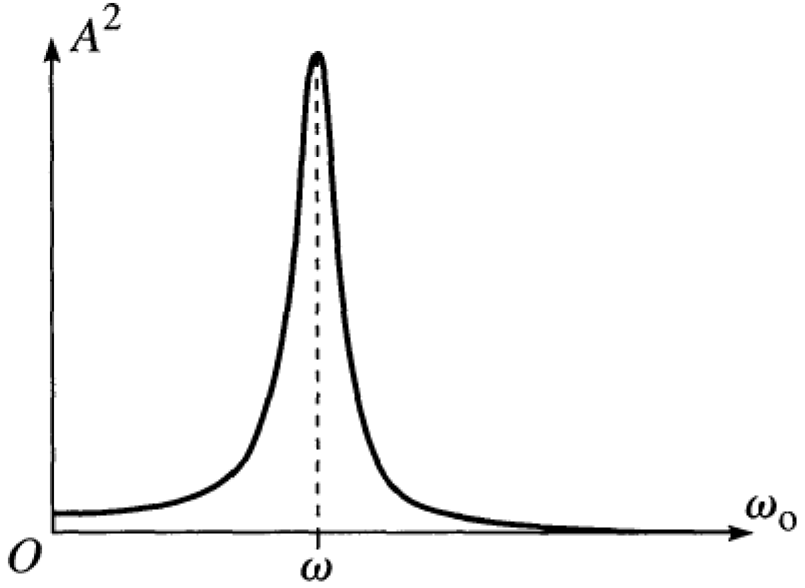
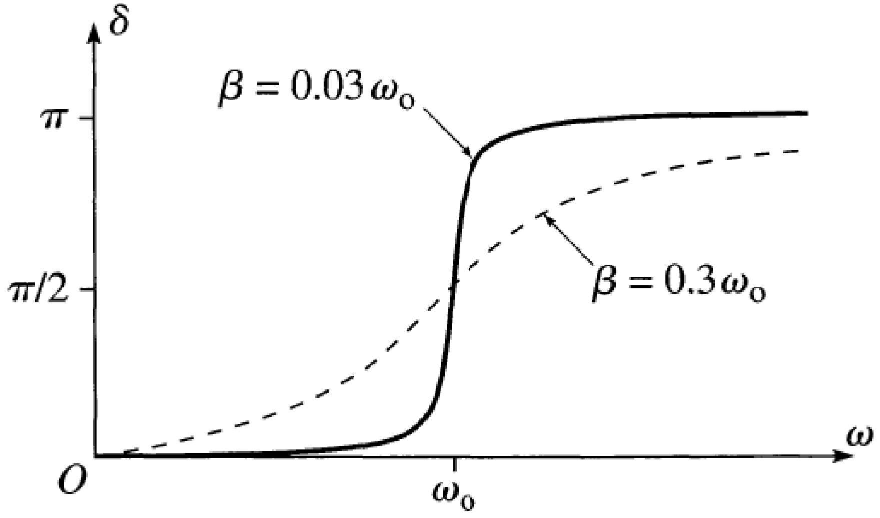


Figure 5.16 The amplitude squared, A^2, of a driven oscillator, shown as a function of the natural frequency ω_o, with the driving frequency ω fixed. The response is dramatically largest when ω_o and ω are close.



$$\delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2}.$$