Physics I

Lecture 12

Reformulation of Newtonian Dynamics

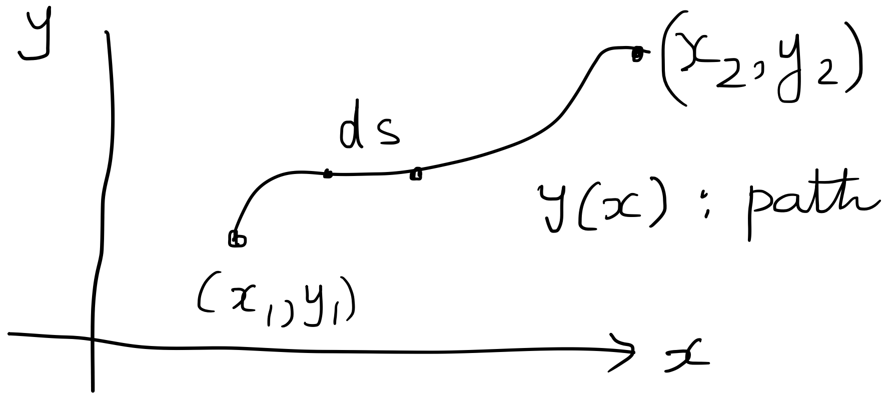
work of Lagrange & Hamilton

several advantages

Some Techniques in Calculus of Variations

Examples

- shortest path between two points in a plane



Task: to find $y = y(x)$ such that it has the shortest length between (x_1, y_1) and (x_2, y_2).

$$ds = \sqrt{dx^2 + dy^2}$$

$$= dx\sqrt{1 + y'^2}$$

$$y' = \frac{dy}{dx}.$$

$$\boxed{L = \int_{x_1}^{x_2} dx \sqrt{1+y'^2}} \rightarrow \text{find } y(x) \text{ such that } L \text{ is minimum}$$

Contrast with elementary calculus, where the unknown is the value of x at a pt. where $f(x)$ is minimum $\quad \frac{df}{dx} = 0$

Ex 2. Fermat's principle in optics.

Path taken by light between two fixed pts
$\longrightarrow$ shortest time

$$\text{time of travel} = \int_1^2 dt = \int_1^2 \frac{ds}{v}$$

If we have single medium $= \frac{1}{c}\int_1^2 n\, ds = \frac{n}{c}\int_1^2 ds$

$\Rightarrow$ same as minimum path

In general, $n = n(x,y)$.

$$\int_1^2 dt = \frac{1}{c}\int_1^2 n(x,y)\, ds = \frac{1}{c}\int_1^2 dx\, n(x,y)\sqrt{1+y'^2}$$

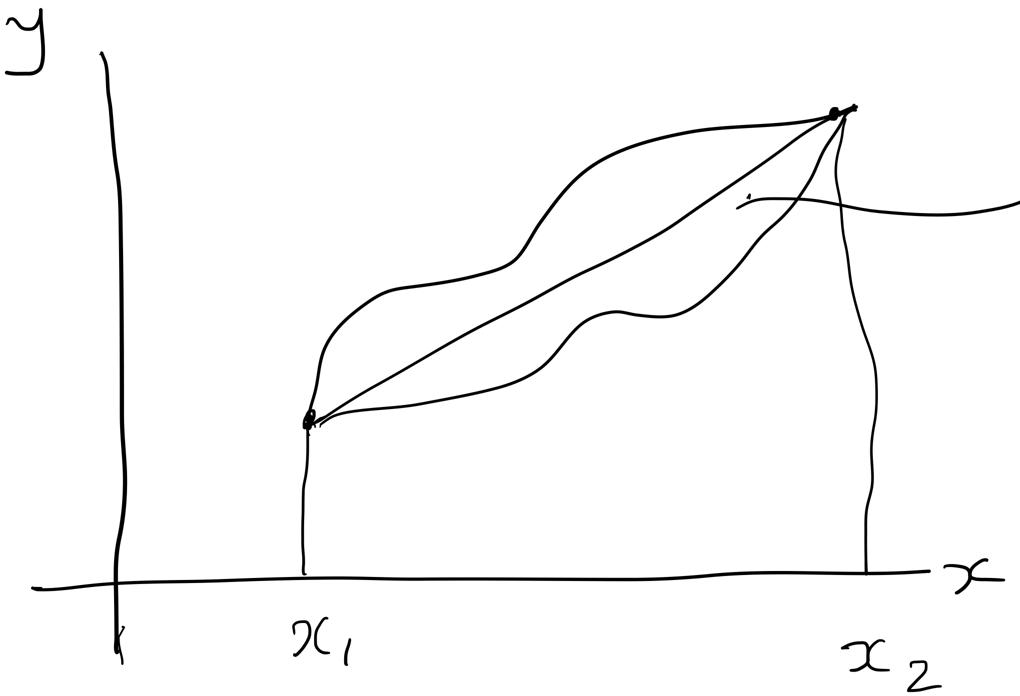
$\rightarrow$ harder problem

Consider

$$J = \int_{x_1}^{x_2} f\left[y(x), y'(x) ; x\right] dx$$

independent variable

$$\boxed{\text{Basic problem: To determine } y(x) \text{ such that } J \text{ is an extremum.}}$$



extremum path $y(x)$

parametric representation

$y = y(\alpha, x)$ such that

$y = y(0, x) = y(x)$

$$y(\alpha, x) = y(0, x) + \alpha \eta(x) \quad \text{—— (1)}$$

continuous first derivative and vanishes at end pts

$\eta(x_1) = \eta(x_2) = 0$

Notice J is now a fn. of α

$$J(\alpha) = \int_{x_1}^{x_2} f\,\{y(\alpha, x), y'(\alpha, x)\,; x\}\,dx \qquad (2)$$

Condition that integral have a stationary value

$$\boxed{\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0} \qquad (3)$$

$\longrightarrow$ necessary condition.

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} f(y, y'; x)\, dx$$

Because limits are fixed differentiation affects only the integrand

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx. \quad \text{--- } 3$$

Recall

$$y = y(0, x) + \alpha \eta(x)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x), \quad \frac{\partial y'}{\partial \alpha} = \eta'(x) \quad \text{--- } 4$$

Plug 4 into 3

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right) dx .$$

Integrate 2nd term by parts

$$\int u\,dv = uv - \int v\,du .$$

$$\int \frac{\partial f}{\partial y'} \frac{d\eta}{dx} dx = \frac{\partial f}{\partial y'} \eta \Big|_{x_1}^{x_2} - \int \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \eta(x) dy$$

$\Leftarrow 0$ η vanishes at limits

$$\int \frac{\partial f}{\partial y'} \frac{d\eta}{dx}\, dx = -\int \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \eta(x)\, dx \quad — (5)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x)\right] dx$$

Using (5)

$$= \int_{x_1}^{x_2} \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}_{=0} \eta(x)\, dx \quad — (6)$$

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

integrand vanishes since $\eta(x)$ is arbitrary.

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \Rightarrow \text{Euler Lagrange eqn.}$$

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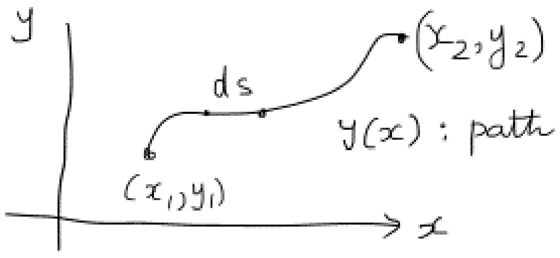

Task: to find $y = y(x)$ such that it has the shortest length between (x_1, y_1) and (x_2, y_2).

$$ds = \sqrt{dx^2 + dy^2}$$
$$= dx\sqrt{1 + y'^2}$$

$y' = \frac{dy}{dx}$. $\longrightarrow$ $L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$. — 1

Compare 1 & 2

$$J = \int_{x_1}^{x_2} f(y, y'; x)\, dx \quad \text{— 2}$$

$$f = \sqrt{1 + y'^2}$$

E-L eq^n:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad f = \sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = \text{const} = c$$

$$y'^2(1-c^2) = c^2$$

$$y' = \pm\frac{c}{\sqrt{1-c^2}} = a$$

$$\boxed{y = ax + b}$$

$y = y_1$
$x = x_1$
$y = y_2$
$x = x_2$

Johann Bernoulli posed the problem of the brachistochrone to the readers of *Acta Eruditorum* in June, 1696.[5][6] He said:

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise

Bernoulli wrote the problem statement as:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.