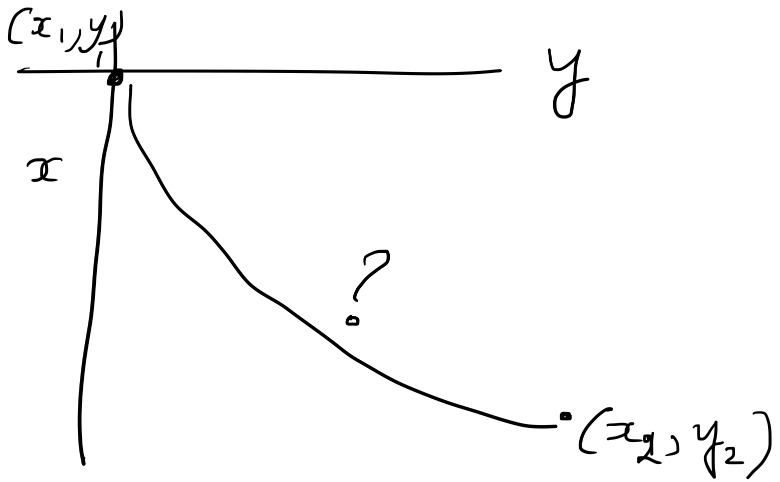
Physics I

Lecture 13

Brachistochrone problem



Path of minimum time between $(x_1, y_1) \longrightarrow (x_2, y_2)$ under gravity? released from rest.

$$E = \frac{1}{2}mv^2 - mgx = 0$$

$$\frac{1}{2}mv^2 = mgx$$

$$v = \sqrt{2gx}$$

$$t = \int_{(x_1, y_1)}^{(x_2, y_2)} \frac{ds}{v}$$

$$= \int_{x_1}^{x_2} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \int_{x_1}^{x_2} dx \frac{\sqrt{1 + y'^2}}{\sqrt{2gx}}.$$

$\sqrt{2g}$ does ~~do~~ not affect final eqn. $\longrightarrow$ $f = \left(\frac{1 + y'^2}{x}\right)^{1/2}$

$$E\text{-}L = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \quad \text{Not } \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$

or $\left(\frac{\partial f}{\partial y'}\right) = \text{const} = (2a)^{-1/2}$.

$f = \left(\frac{1+y'^2}{x}\right)^{1/2}$

$$\frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \quad \text{— (1)}$$

from (1) $\quad y = \int \frac{x\,dx}{(2ax-x^2)^{1/2}} \quad \text{— (2)}$

$$\left.\begin{aligned} x &= a(1-\cos\theta) \\ dx &= a\sin\theta\, d\theta \end{aligned}\right\} \quad \text{— (3)}$$

$$x = a(1-\cos\theta) \;,\; dx = a\sin\theta\, d\theta \quad \Big\} \text{—} (3)$$

Substituting (3) in (2)

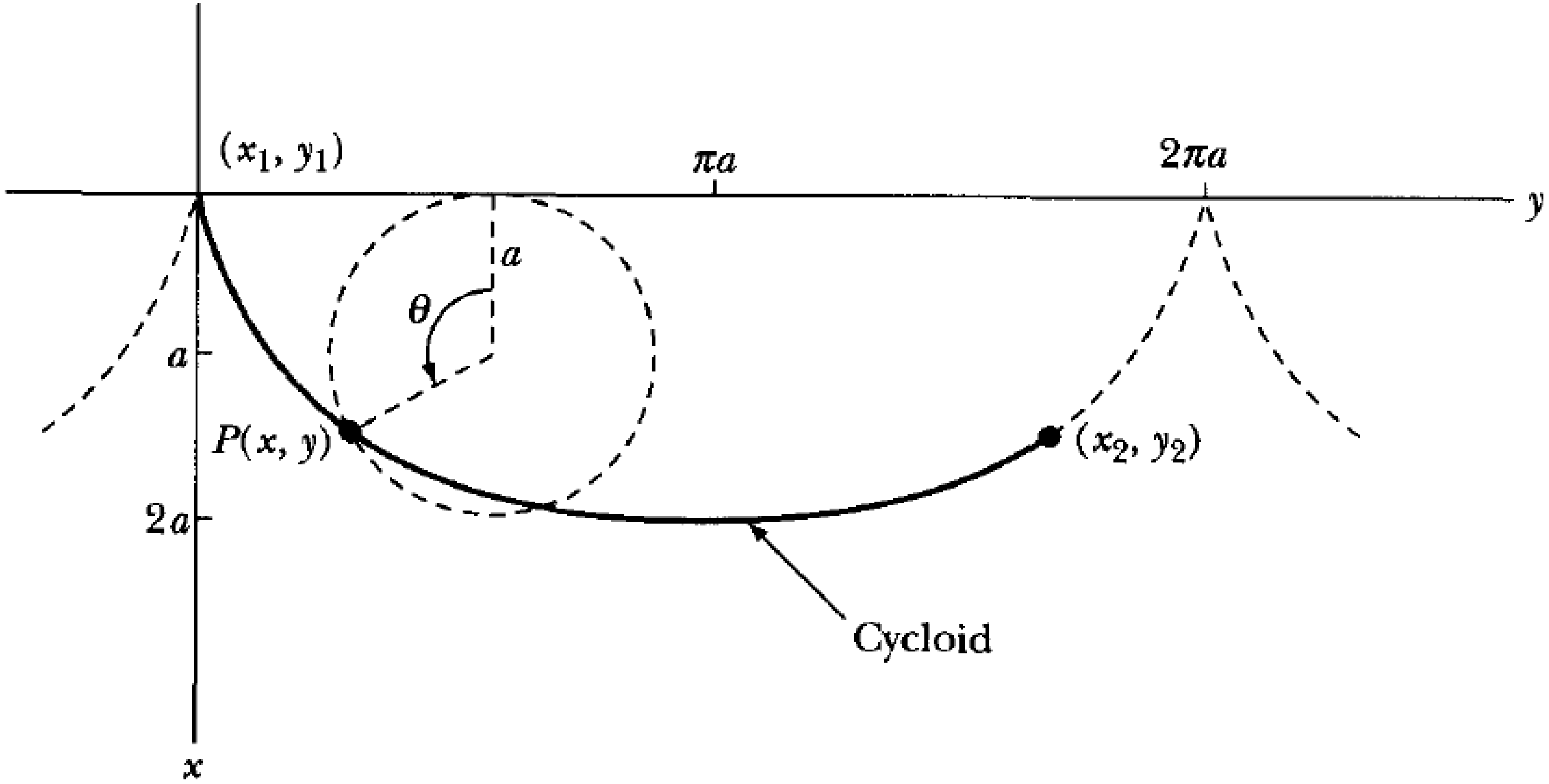
$$y = a\int (1-\cos\theta)\, d\theta = a(\theta - \sin\theta) + \text{const}.$$

$$\boxed{\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}}$$

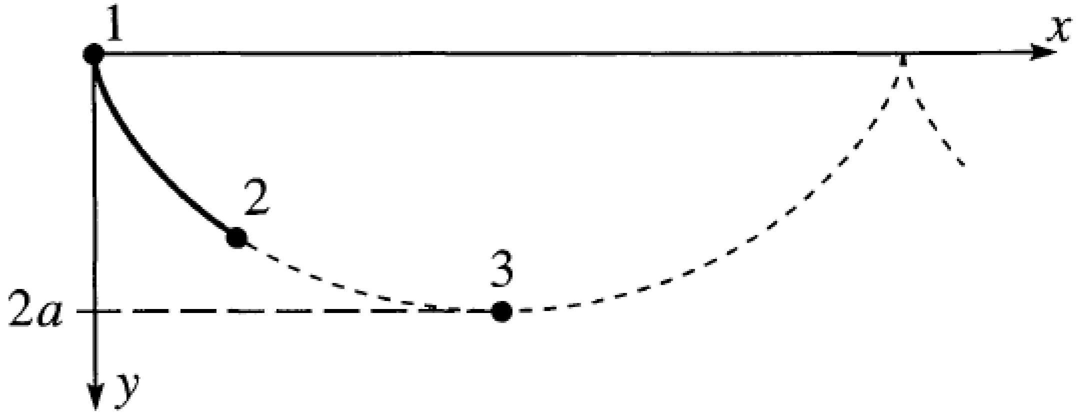
const = 0

$(0,0)$ starting pt.

a has to be adjusted to allow curve to pass through (x_2, y_2).



(x_1, y_1)
πa
$2\pi a$
y
a
θ
a
$P(x, y)$
(x_2, y_2)
$2a$
Cycloid
x



Time period independent of amplitude
isosynchronous.