Physics I

Lecture 14

Recap.

$$J = \int_{x_1}^{x_2} f\left[y(x), y'(x); x\right] dx$$

the path that extremezies it, $y(x)$

is determined by the E-L eqn.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

Generalization to multivariable case

$$f\left[y_1(x), y_1'(x), y_2(x), y_2'(x) \cdots ; x\right]$$

$$f\left[y_i(x), y_i'(x) ; x\right] \qquad i = 1 \cdots n$$

$$y_i(x) = y_i(0,x) + \alpha \eta_i(x)$$

following 1 variable derivation

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right) \eta_i(x)\, dx$$

η_i's are independent $\quad \partial J / \partial \alpha = 0$

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \longrightarrow \text{E-L eqns}$$

Hamilton's Principle & Lagrangian Dynamics

- experience shows that in inertial frames particle motion is correctly described by Newton's Laws
  $\vec{F} = \dot{\vec{p}}$

- Practical difficulties in applying Newton's Laws.
  e.g. non Cartesian coordinates $\rightsquigarrow$ motion on a sphere
  projection of vector eqns on the sphere is complicated

- Constraints, example bead sliding on wire
  forces of constraint are complicated and occasionally cannot get explicit expressions. $\vec{F}$ includes all forces.

Alternative formulation of Newtonian dynamics

Lagrange Eqns.

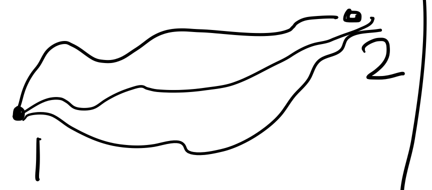
$\updownarrow$

$\boxed{\text{Hamilton's Principle}} \equiv$ Newton's Laws.

elegant, applicable to a wide variety of physical phenomena including field

We will stick to conservative systems.

Hamilton's Principle



The actual path which a particle follows between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} L \, dt$$

is stationary when taken along the actual path

where $L = T - U$; L: Lagrangian

T: kinetic energy

U: potential energy.

Summary

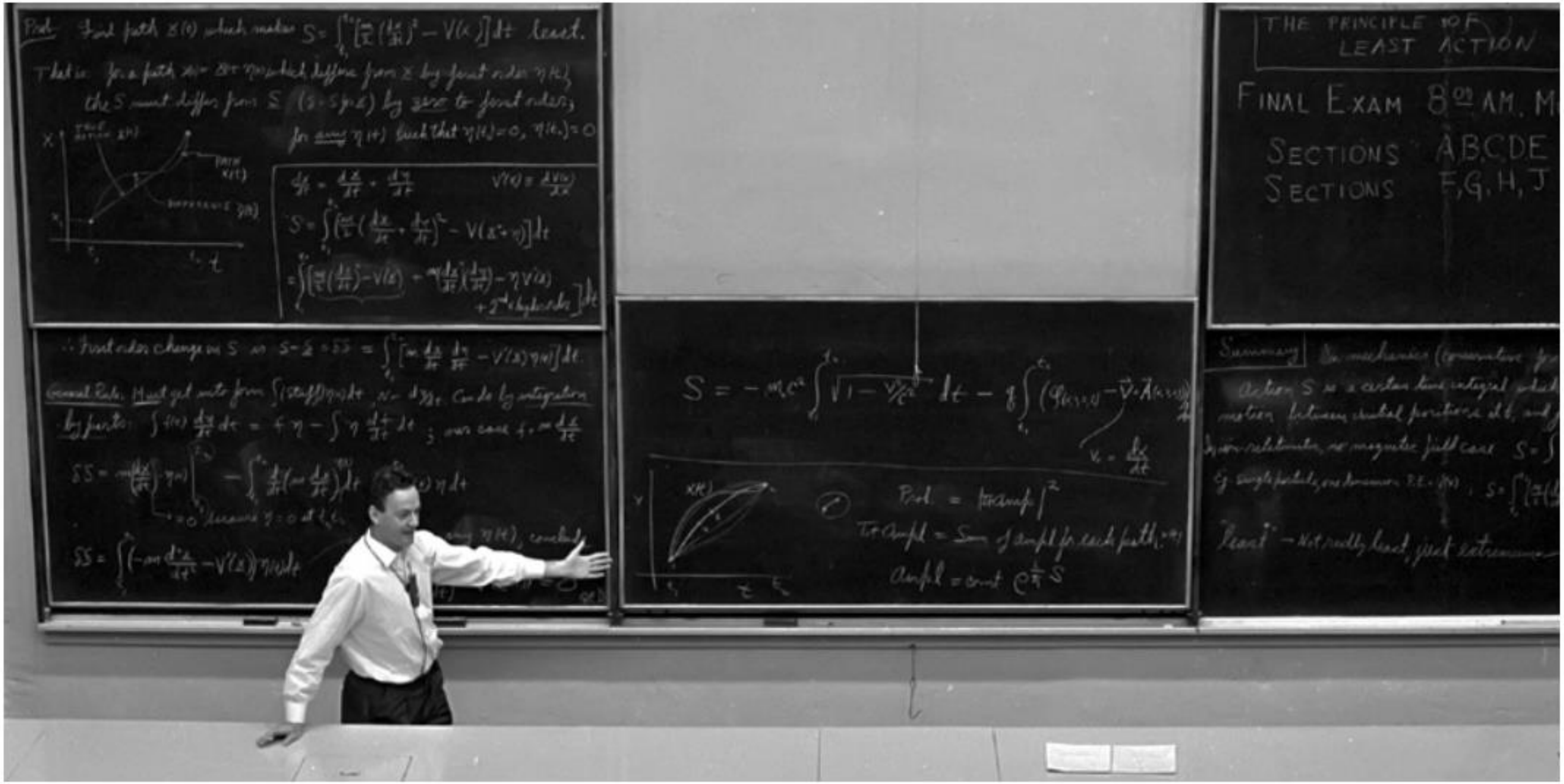
In mechanics (conservative forces only):

Action S is a certain time integral which is "least*" for the true motion between initial positions at t_1 and final ones at t_2.

In non-relativistic, no magnetic field case $S = \int (\text{Kinetic Energy} - \text{Potential Energy})\,dt$ NOT +

Eg. single particle, one dimension P.E. = $V(x)$; $S = \int_{t_1}^{t_2} \left[\frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 - V(x(t))\right] dt.$

"least*" → Not really least, just extremum ← means first order change = 0.


THE PRINCIPLE OF LEAST ACTION
FINAL EXAM 8 AM
SECTIONS ABCDE
SECTIONS F,G,H,J
Summary

$$S = \int_{t_1}^{t_2} L(x_i, \dot{x}_i; t)\, dt \quad \text{--- (1)}$$

Recall $$J = \int_{x_1}^{x_2} f[y_i, y_i'; x]\, dx \quad \text{--- (2)}$$

make the correspondence

$$\left.\begin{aligned} x &\longrightarrow t \\ y_i(x) &\longrightarrow x_i(t) \\ y_i'(x) &\longrightarrow \dot{x}_i(t) \end{aligned}\right\}$$

$$\boxed{\begin{array}{c}\text{Euler-Lagrange eqn.} \\ \dfrac{\partial L}{\partial x_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_i}\right) = 0\end{array}}$$

Examples

① Free particle in 3D.

$$L = T - U \quad , \quad U = 0 .$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) .$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0$$

$$\frac{d}{dt}(m\dot{x}) = 0 = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}(m\dot{z})$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0 .$$

Ex 2 1-d Harmonic Oscillator

$$L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2 .$$

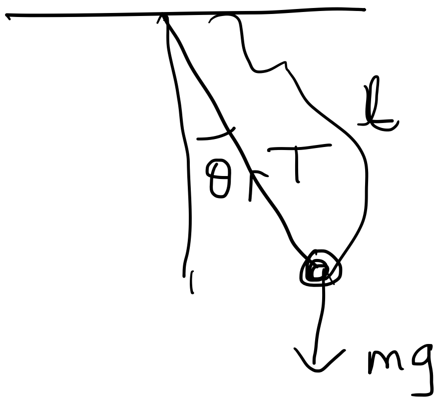
$$\frac{\partial L}{\partial x} = -kx \quad ; \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 .$$

$$m\ddot{x} + kx = 0$$

Ex 3

Plane pendulum



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta).$$

E-L eqn.

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$ → identical to Newtonian result.

Generalized Coordinates

Degrees of freedom : In general for n (free) particles is $3n$.

m constraints

degrees of freedom $\boxed{s = 3n - m}$

Need to choose s coordinates to describe motion

need not be Cartesian coordinates (can choose curvilinear coordinates, spherical, cylindrical)

need not even have dimensions of length.

generalized coordinates $\{q_i\} \rightarrow$ not unique generalized coordinates $\{q_j\}$ + generalized velocities $\{\dot{q}_j\}$

Coordinate transformations

$$\boxed{\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, \cdots q_s, t) \\ &= x_{\alpha,i}(q_j, t) \\ \dot{x}_{\alpha,i} &= \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \end{aligned}}$$

where $\alpha = 1, \cdots n$

$i = 1, 2, 3$

$j = 1, \cdots s$.

$m = 3n - s$.

Inverse transform

$$\left.\begin{aligned} q_j &= q_j(x_{\alpha,i}, t) \\ \dot{q}_j &= \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{aligned}\right\}$$

+ constraint eqn

$f_k(x_{\alpha,i}, t) = 0$

$k = 1, \cdots m$