Physics I

Lecture 15

Hamilton's Principle

$$S = \int_{t_1}^{t_2} L \, dt$$

$$\boxed{\delta S = 0}$$

$$\delta S \equiv \frac{\partial S}{\partial \alpha} d\alpha$$

$$L = T - U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \implies \text{E-L eqn.}$$

generalized coordinates $\{q_j\}$ + generalized velocities $\{\dot{q}_j\}$

Coordinate transformations

$$\boxed{\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, \cdots q_s, t) \\ &= x_{\alpha,i}(q_j, t) \\ \\ \dot{x}_{\alpha,i} &= \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \end{aligned}}$$

where $\alpha = 1, \cdots n$

$i = 1, 2, 3$

$j = 1, \cdots s$.

$m = 3n - s$.

Inverse transform

$$\left.\begin{aligned} q_j &= q_j(x_{\alpha,i}, t) \\ \dot{q}_j &= \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{aligned}\right\}$$

+ constraint eq$^{\underline{n}}$

$f_k(x_{\alpha,i}, t) = 0$

$k = 1, \cdots m$

Lagrangian is a scalar $\Rightarrow$ coordinate invariant

$$L = T(\dot{x}_{\alpha,i}) - U(x_{\alpha,i}) = T(q_j, \dot{q}_j, t) - U(q_j, t)$$

$$L = L(q_j, \dot{q}_j, t) \Rightarrow \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0$$

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0} \quad j = 1, 2, \cdots s.$$

Is the Lagrangian unique? What degree of arbitrariness does it have?

for example U is not unique $U \to U + \text{constant}$

$\Longrightarrow$ does not change eqns of motion.

It turns out that L is arbitrary upto

$$L' \longrightarrow L + \frac{d}{dt} f(q_i, t)$$

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt + \int_{t_1}^{t_2} \frac{df}{dt}\, dt$$

$$= S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1).$$

$$S' = S + \underbrace{f(q^{(2)}, t_2) - f(q^{(1)}, t_1)}.$$

$\to$ does not vary on variation, end pts fixed.

$\downarrow$ notice

$$\delta S' = \delta S$$

$\downarrow$

Leads to identical E-L eqns of motion.

Equivalence of Lagrange & Newton's eqns

choose generalized coordinates as Cartesian coordinates

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \qquad i = 1, 2, 3$$

$$\frac{\partial (T-U)}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{x}_j}\right) = 0 .$$

For conservative system in rectangular coordinates

$T = T(\dot{x}_i)$ and $U = U(x_i)$.

$$\therefore \frac{\partial T}{\partial x_i} = 0 \quad , \quad \frac{\partial U}{\partial \dot{x}_i} = 0$$

$$\frac{\partial}{\partial x_j}(T-U) - \frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{x}_j}\right] = 0 \quad \Big\} \quad T = \sum_{j=1}^{3} \frac{1}{2} m \dot{x}_i^2$$

$$\frac{\partial T}{\partial \dot{x}_j} = m \dot{x}_i$$

$$-\frac{\partial U}{\partial x_j} - \frac{d}{dt}\left[m \dot{x}_j\right] = 0$$

But we know $-\dfrac{\partial U}{\partial x_j} = F_j$

$$\boxed{m \ddot{x}_j = F_j} \Longrightarrow \text{recover Newton's Law.}$$

Application of Lagrangian formulation

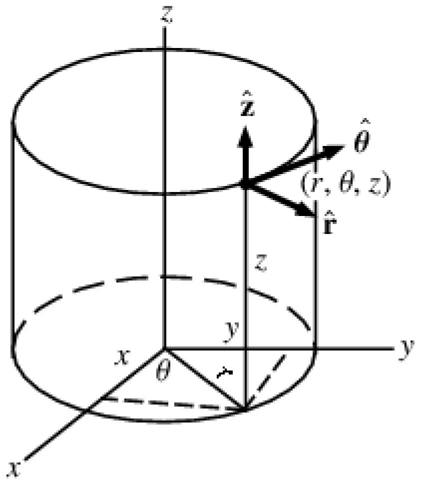
1. Free particle . $V = 0$

$$L = \frac{1}{2} m v^2$$

choose generalized coordinates (x, y, z) Rectangular

Cartesian Coordinates

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$



Cylindrical coordinates.

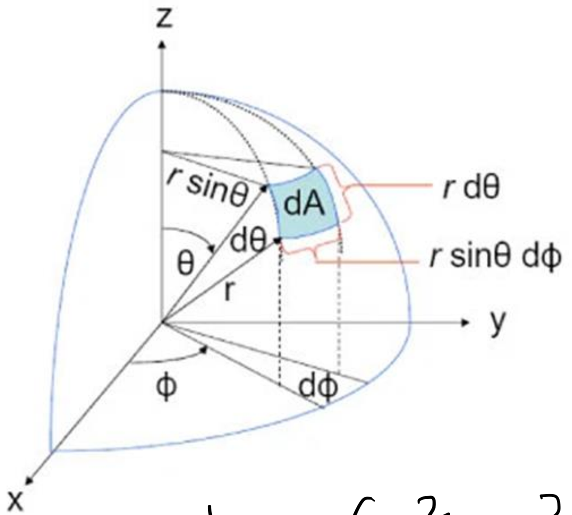
$$\vec{r} = (r, \theta, z)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$



Spherical Polar coordinates

$$\vec{r} = (r, \theta, \phi)$$

$$x = r\sin\theta\cos\phi$$

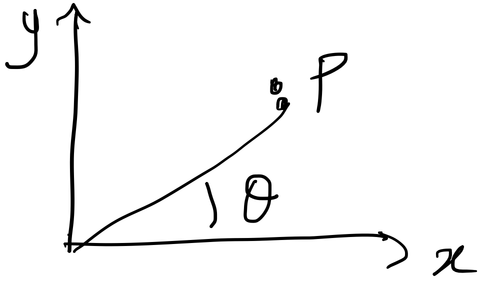
$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta.$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

2D Polar coordinates

free particle



$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

E-L eqns. (r, θ)

x, y

$m\ddot{x} = 0$

$m\ddot{y} = 0$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt}(m\dot{r}) - m r \dot{\theta}^2 = 0$$

$$\boxed{m\ddot{r} = m r \dot{\theta}^2} \quad \text{—} (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0 \quad \text{—} (2)$$

L independent of θ $\longrightarrow$ $\boxed{m r^2 \dot{\theta} = \text{const}}$

Ang mom conserved.

Conservation Laws

cyclic coordinate.

Lagrangian is independent of this coordinate

q_k

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

$\rightarrow 0$; q_k cyclic.

In our previous ex θ was cyclic

Ang mom conserved

$$\boxed{\frac{\partial L}{\partial \dot{q}_k} = \text{conserved}}.$$

Generalized momentum is conserved.

$$\frac{\partial L}{\partial \dot{q}_k}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

2D (x, y)

$\equiv$ Cartesian coordinates

cyclic coordinates ?

$\downarrow$ (x, y)

both cyclic.

E-L eqn.

$\downarrow$

Generalized mom: x $\quad \frac{\partial L}{\partial \dot{x}} = \underbrace{m\dot{x}}_{p_x} \longrightarrow$ conserved

Gen. mom for y $\quad \frac{\partial L}{\partial \dot{y}} = \underbrace{m\dot{y}}_{p_y} \longrightarrow$ conserved.