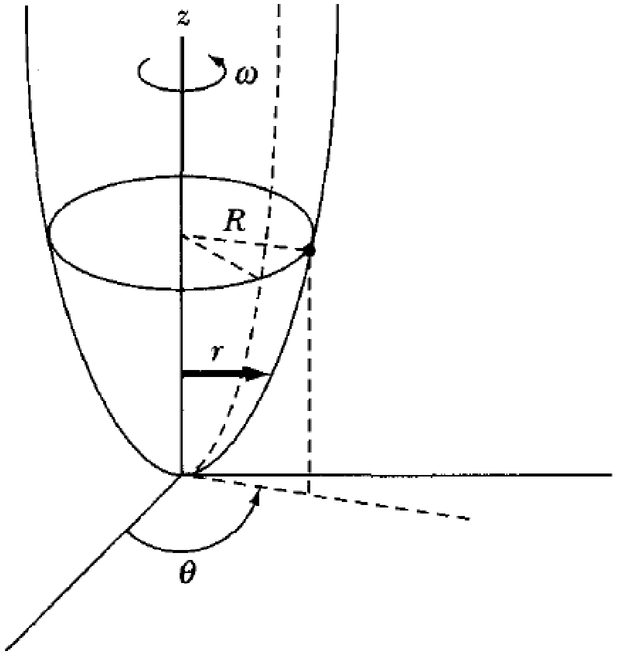
Physics I

Lecture 16

Recap

- E-L eqns $\equiv$ Newton's Laws

- If q_K is a cyclic coordinate, the corresponding $\underbrace{\text{generalized momentum}}_{\frac{\partial L}{\partial \dot{q}_K}}$ is conserved.



A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$. The bead rotates in a circle of radius R when the wire is rotating about its vertical with angular vel. ω. Find the value of c

generalized coordinates
(r, θ, z)

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right]$$

$U = 0, \; z = 0$

$U = mgz$.

$z = cr^2$.

$\dot{z} = 2c\dot{r}r$

$\theta = \omega t$. $\dot{\theta} = \omega$.

$$L = T - U$$

$$= \frac{m}{2}\left[\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2\right] - mgz \, .$$

Plug in the constraints

$$= \frac{m}{2}\left[\dot{r}^2 + 4c^2 r^2 \dot{r}^2 + r^2\omega^2\right] - mgcr^2 \, .$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}\left(2\dot{r} + 8c^2 r^2 \dot{r}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{m}{2}\left(2\ddot{r} + 16c^2 r \dot{r}^2 + 8c^2 r^2 \ddot{r}\right) \quad \text{—— } (1)$$

$$\frac{\partial L}{\partial r} = m\left(4c^2 r \dot{r}^2 + r\omega^2 - 2gcr\right) \quad \text{—— } (2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

Plugging in from 1 & 2

$$\ddot{r}\left(1 + 4c^2r^2\right) + \dot{r}^2\left(4c^2r\right) + r\left(2gc - \omega^2\right) = 0$$

But $r = R$, kills $\ddot{r}$, $\dot{r}$ terms.

$$R\left(2gc - \omega^2\right) = 0$$

$$\boxed{c = \frac{\omega^2}{2g}}$$

New look at Conservation Laws

A Theorem concerning K.E (n particles in 3D)

K.E in fixed rectangular coordinates.

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2 \quad \text{--- (3)}$$

Now let us transform to generalized coordinates and velocities.

m constraints $m = 3n - s$

$s = 3n - m$

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t) \qquad j = 1, \dots s. \quad \text{--- (4)}$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t} \quad \text{--- (5)}$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t} \quad \text{— (5)}$$

Plug (5) into (3)

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2 \quad \text{— (3)}$$

$$\dot{x}_{\alpha,i}^2 = \sum_{j,k} \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k + 2 \sum_j \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j$$

$$+ \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \quad \text{— (6)}$$

$$T = \sum_\alpha \sum_{i,j,k} \overbrace{\frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k}} \dot{q}_j \dot{q}_k + \sum_{\alpha,i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j$$

$$+ \sum_\alpha \sum_i \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \quad \text{— (7)}$$

Can rewrite (7) as.

$$T = \sum_{j,k} a_{jk}\, \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c \quad \text{— (8)}$$

Special case, when the transformation does not explicitly depend on time

$$\frac{\partial x_{\alpha,i}}{\partial t} = 0 \quad \text{— (9)} \implies b_j = 0, \; c = 0$$

Under these conditions, kinetic energy is a homogeneous quadratic fn. of the generalized velocities.

Note that

$a_{jk} = a_{kj}$

$$\boxed{T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k} \quad (10)$$

Differentiate T w.r.t $\dot{q}_l$, [Are you familiar with $\delta_{ij} = ?$]

$$\boxed{\begin{aligned} \delta_{ij} &= 0 \quad i \neq j \\ &= 1 \quad i = j \end{aligned}}$$

Note . $\boxed{\dfrac{\partial \dot{q}_j}{\partial \dot{q}_l} = \delta_{jl}}$

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_{j,k} a_{jk} \delta_{jl} \dot{q}_k + \sum_{j,k} a_{jk} \dot{q}_j \delta_{kl}$$

$$= \sum_k a_{lk} \dot{q}_k + \sum_j a_{jl} \dot{q}_j$$

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_k a_{lk}\dot{q}_k + \sum_j a_{jl}\dot{q}_j$$

k, j are dummy indices

$$= \sum_k a_{lk}\dot{q}_k + \sum_k a_{lk}\dot{q}_k \qquad [a_{jl} = a_{lj}]$$

$$= 2\sum_k a_{lk}\dot{q}_k$$

$$\boxed{\sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2\sum_{l,k} a_{lk}\dot{q}_k\dot{q}_l = 2T} \quad (11)$$

special case of Eulers theorem, $f(y_k)$ is a homogeneous fn. of y_k of degree n

$$\sum_k y_k \frac{\partial f}{\partial y_k} = nf$$