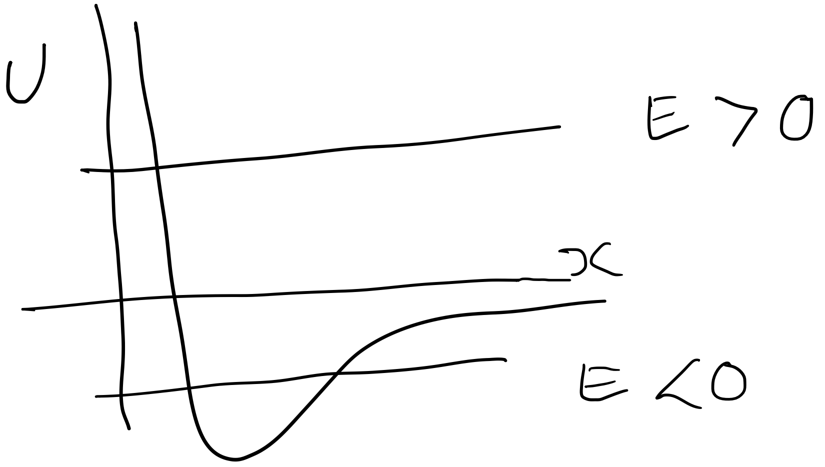
Physics 1

Lecture 17

- Always plot the potential



- In problems like
$$U(x) = \frac{\alpha}{x} - \frac{\beta}{x^2}$$
always convenient to take ∞ as reference pt.

- In problem 5 $F_x = ayz + bx + c$, $F_y = axz + bz$

$F_z = \ldots$

$$\vec{\nabla} \times \vec{F} = 0$$

$$U = -\int \vec{F} \cdot d\vec{r} = \int F_x\,dx + \int F_y\,dy + \int F_z\,dz$$

Remember, line integral.

$$W_a = \int_a \mathbf{F}\cdot d\mathbf{r} = \int_O^Q \mathbf{F}\cdot d\mathbf{r} + \int_Q^P \mathbf{F}\cdot d\mathbf{r} = \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,y)\,dy$$

$$= 0 + 2\int_0^1 dy = 2.$$

$$\mathbf{F} = (y, 2x)$$

example

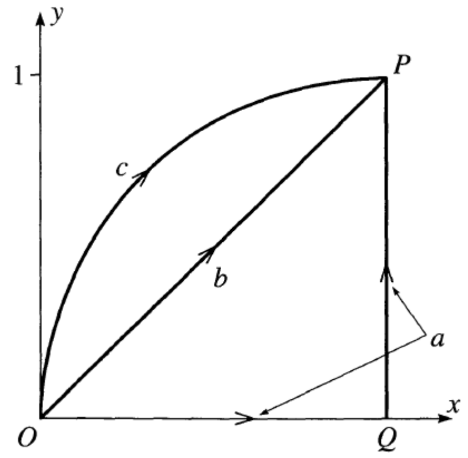


Figure 4.2 Three different paths, a, b, and c, from the origin to the point $P = (1, 1)$.

On the path b, $x = y$, so that $dx = dy$, and

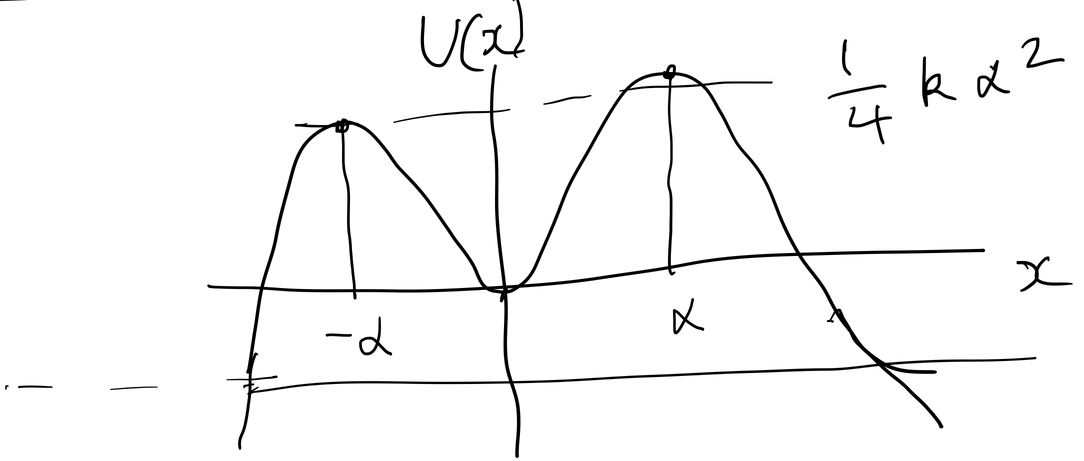
$$W_b = \int_b \mathbf{F}\cdot d\mathbf{r} = \int_b (F_x\,dx + F_y\,dy) = \int_0^1 (x + 2x)dx = 1.5.$$

- $\vec{F} = \frac{a}{r}\hat{r}$ (a, b, c) are constants.

$F_x = \frac{a^2}{r}$, $F_y = \frac{ab}{r}$, $F_z = \frac{ac}{r}$ X .

Problem 2

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}$$

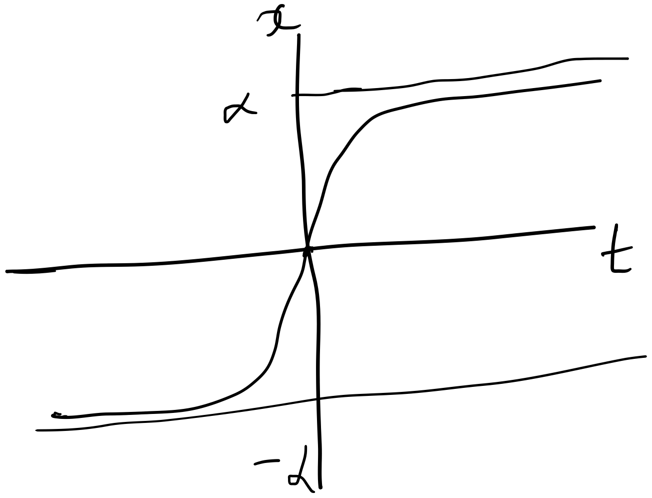


$\frac{1}{4}k\alpha^2 = E$; Are $\alpha, -\alpha$ turning pts?

$E = U$ at these pts.

not turning pts.

pts of unstable equi.



$$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}\frac{kx^4}{\alpha^2}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{4}\frac{kx^4}{\alpha^2}.$$

$$E = \frac{1}{4}k\alpha^2.$$

$$\dot{x}^2 = \frac{1}{4}\frac{kx^4}{\alpha^2} - \frac{1}{2}kx^2 + \frac{1}{4}k\alpha^2.$$

$$\sqrt{\frac{2m}{k}}\int\frac{dx}{(\alpha^2-x^2)} = \int dt$$ can be exactly

$$\hookrightarrow x = \alpha\tanh\left(\sqrt{\frac{k}{2m}}\,\alpha t\right)$$

Recap.

$$\boxed{\sum_{\ell} \dot{q}_{\ell} \frac{\partial T}{\partial \dot{q}_{\ell}} = 2T}$$

provided $x_{\alpha,i} = x_{\alpha,i}(q_j, \cancel{t})$

Conservation Laws & Symmetries

time is homogeneous within an inertial coordinate system. $\longrightarrow$ symmetry

$\therefore$ **Lagrangian of a closed system cannot depend explicitly on time**

$$\frac{\partial L}{\partial t} = 0 \quad \text{—— (1)} \qquad L(q_j, \dot{q}_j; t)\,.$$

$$\frac{dL}{dt} = \sum_{j=1}^{s} \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_{j=1}^{s} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \cancelto{0}{\frac{\partial L}{\partial t}} \quad \text{—— (2)}$$

Use Euler-Lagrange eqn. in (2)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{—— (3)}$$

$$\frac{dL}{dt} = \sum_{j} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_{j} \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \quad \text{—— (4)}$$

Can rewrite (4) as

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) \quad — (5)$$

or $$\frac{d}{dt}\left\{\underbrace{L - \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j}_{-H}\right\} = 0 .$$

Hamiltonian

$$\Rightarrow \quad \frac{dH}{dt} = 0 \qquad \boxed{H = \text{const}} \quad — (6)$$

If the potential energy $U(x)$ does not explicitly on the velocities $\dot{x}_{\alpha,i}$ or t,

$$U = U(x_{\alpha,i})$$

the coordinate transformations will be of the form

$$x_{\alpha,i} = x_{\alpha,i}(q_j) \quad \text{or} \quad q_j = q_j(x_{\alpha,i}).$$

$$U = U(q_j), \quad \frac{\partial U}{\partial \dot{q}_j} = 0$$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \quad \text{—— (7)}$$

So now

$$-H = L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}$$

$$= (T-U) - \underbrace{\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}}_{2T}$$

$$= T - U - 2T = -(T+U).$$

$$= -E$$

$\boxed{H = E}$ $\longrightarrow$ energy conserved.

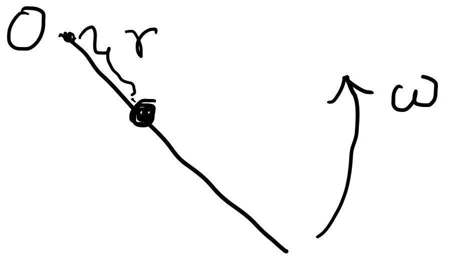
$H = E$ only if certain conditions are met

1. The eqns of transfn of coordinate must be independent.

2. The potential must be independent of velocity $\longrightarrow \dfrac{\partial U}{\partial \dot{q}_j} = 0$

Bead on Stick:

A stick is pivoted at the origin and is arranged to swing around in a horizontal plane with constant angular speed ω. A bead of mass m slides frictionlessly along the stick. Let r be the radial position of the bead. Find the Hamiltonian. Explain why this is **not** the energy of the bead.



No potential energy, only K.E

$$\dot{\theta} = \omega$$

$$L = T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \omega^2 .$$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

$$= \frac{\partial L}{\partial \dot{r}} \dot{r} - \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \omega^2 .$$

$$H = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2 \neq E$$

$$\frac{\partial L}{\partial t} = 0$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 .$$