Physics I

Lecture 18

Homogeneity of time $\longrightarrow$ $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \quad H = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad \Rightarrow \text{conserved}$$

when $x_{i,\alpha} = x_{i,\alpha}(q_j)$ not time dependent

$$H = E$$

Closed inertial system

Homogeneity of space $\longrightarrow$ all points of space are equivalent

$\hookrightarrow$ The Lagrangian of the system is invariant under a translation of the entire system in space.

$$\vec{r}_\alpha \Rightarrow \vec{r}_\alpha + \underbrace{\delta\vec{r}_\alpha} = \vec{r}_\alpha + \vec{\epsilon} \quad \text{—— (1)}$$

clearly $\delta\dot{\vec{r}}_\alpha = 0$

$$\delta L = \sum_\alpha \sum_i \frac{\partial L}{\partial x_{\alpha,i}} \cdot \delta x_{\alpha_i} + \sum_\alpha \sum_i \frac{\partial L}{\partial \dot{x}_{\alpha,i}} \underbrace{\delta \dot{x}_{\alpha,i}}_{=0}$$

$$= \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta\vec{r}_\alpha \quad \text{—— (2)}$$

$$\delta L = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta \vec{r}_\alpha .$$

$$= \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot \vec{\epsilon}$$

$$= \vec{\epsilon} \cdot \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} .$$

If L is invariant under the transfn. $\delta L = 0$

$$\delta L = 0 \Rightarrow \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

$\vec{\epsilon}$ arbitrary

$L = T - U$

$$\equiv \sum_\alpha \vec{F}_\alpha = 0$$

$$\sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

E-L equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) - \frac{\partial L}{\partial \vec{r}_\alpha} = 0$

$$\rightarrow \sum_\alpha \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) = 0 \quad , \quad \frac{d}{dt} \sum_\alpha \Big(\underbrace{\frac{\partial L}{\partial \vec{v}_\alpha}}_{\vec{p}_\alpha}\Big) = 0 .$$

$$\frac{d}{dt}\left(\sum_\alpha \vec{p}_\alpha\right) = 0$$

$$\Longrightarrow \boxed{P = \sum_\alpha \vec{p}_\alpha} \text{ consenved}$$

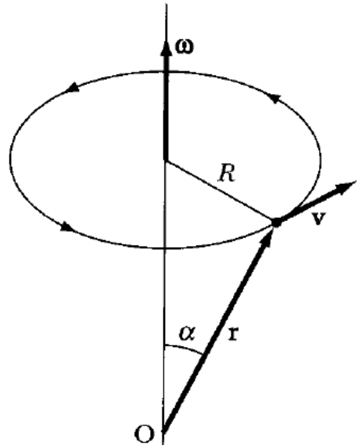
Generalized coordinates

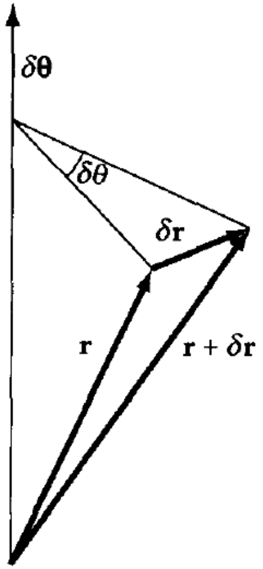
$$\sum_{\alpha, i} \frac{\partial L}{\partial \dot{q}_{i,\alpha}} = \text{consenved}$$

Next symmetry

Isotropy of space.

Lagrangian invariant under rotations.





A particle moving arbitrarily in space, can always be considered, *at a given instant,* to be moving in a plane circular path about a certain axis. That is, the path that a particle describes during an infinitesimal time interval δt is represented by an infinitesimal arc of a circle. The line passing through the centre and perpendicular to the instantaneous direction of motion is called the instantaneous axis of rotation.

$$\boxed{\delta\vec{r} = \delta\vec{\theta} \times \vec{r}}$$

$$\omega = \frac{d\theta}{dt} \qquad \dot{\vec{r}} = \vec{v}$$

$$\vec{v} = \vec{\omega} \times \vec{r} \quad ; \quad v = r \sin\alpha\, \omega$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$$

$$\delta\dot{\vec{r}} = \delta\vec{\theta} \times \dot{\vec{r}}$$

E-L

$$\frac{\partial L}{\partial \vec{r}_\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) = \frac{d\vec{p}_\alpha}{dt}$$

$$\delta L = \sum_\alpha \left(\frac{\partial L}{\partial \vec{r}_\alpha} \cdot \delta\vec{r}_\alpha + \frac{\partial L}{\partial \vec{v}_\alpha} \cdot \delta\vec{v}_\alpha \right)$$

$$= \sum_\alpha \left(\dot{\vec{p}}_\alpha \cdot \delta\vec{r}_\alpha + \vec{p}_\alpha \cdot \delta\vec{v}_\alpha \right)$$

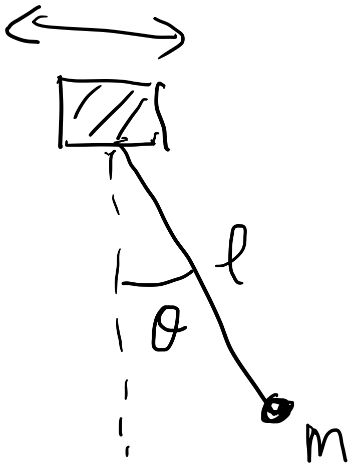
$$= \sum_\alpha \left[\dot{\vec{p}}_\alpha \cdot \delta\vec{\theta} \times \vec{r}_\alpha + \vec{p}_\alpha \cdot \delta\vec{\theta} \times \dot{\vec{r}}_\alpha \right]$$

$$= -\delta\vec{\theta} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha \times \vec{p}_\alpha \right) = 0$$

$$\Rightarrow \frac{d}{dt} \sum_\alpha \left(\vec{r}_\alpha \times \vec{p}_\alpha \right) = 0 \Longrightarrow \boxed{\vec{L} = \text{const}}$$

Noether's Theorem : Every continuous symmetry of a Lagrangian corresponds to a conserved quantity.

A pendulum consists of a mass m and a massless stick of length l. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos\omega t$. What is the general solution for the angle of the pendulum as a function of time ? You are allowed to make a small angle approximation.



Coordinates of mass (X, Y)

$$(X, Y) = (x + l\sin\theta, -l\cos\theta)$$

to find K.E, find V^2.

$$V^2 = \dot{X}^2 + \dot{Y}^2 = l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta$$

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + mgl\cos\theta$$

$$L = \frac{1}{2} m \left(l^2 \dot{\theta}^2 + \dot{x}^2 + 2 l \dot{x} \dot{\theta} \cos\theta \right) + mgl\cos\theta$$

E-L eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(ml^2 \dot{\theta} + ml\dot{x}\cos\theta \right) = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$\Rightarrow l\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta \qquad x = A\cos\omega t$$

$$l\ddot{\theta} - A\omega^2 \cos\omega t \cos\theta + g\sin\theta = 0$$

small angle approx

$$\ddot{\theta} + \omega_0^2 \theta = a\omega^2 \cos\omega t \qquad \omega_0^2 = g/l$$

$$a = A/l$$

$$\ddot{\theta} + \omega_0^2 \theta = a\omega^2 \cos\omega t \longrightarrow \text{Driven oscillator}$$

$$\theta(t) = \underbrace{\frac{a\omega^2}{\omega_0^2 - \omega^2} \cos(\omega t)}_{\text{particular solution}} + \underbrace{C \cos(\omega_0 t + \phi)}_{\text{homogeneous}}$$