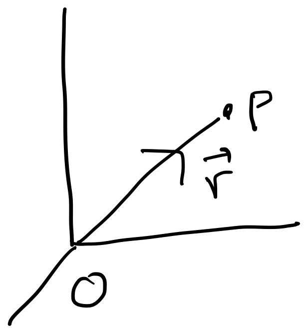
Physics I

Lecture 19

Central Force Dynamics

Motion of a two body system affected by a force along the line joining their centres.

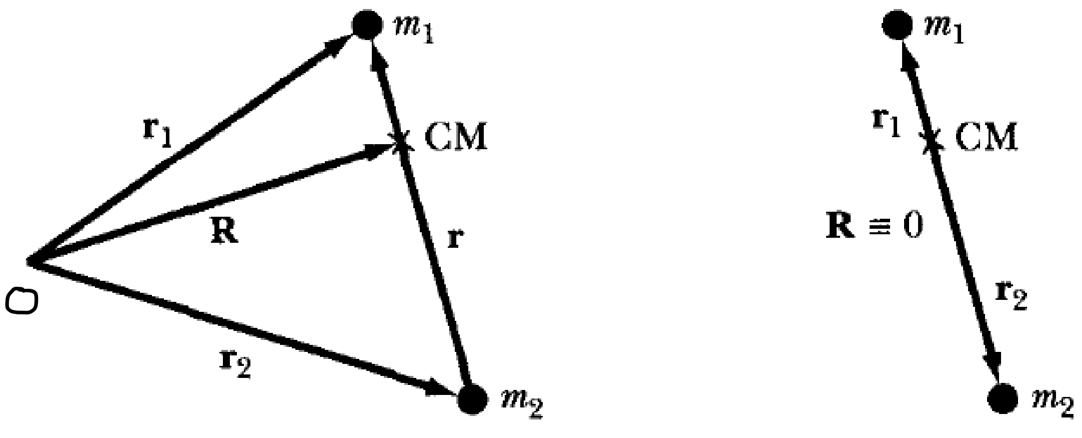
↳ motion of planets, moons, comets, ... Rutherford scattering etc.



$$\vec{F} = F(r)\,\hat{r} \longrightarrow \text{Conservative force.}$$

↳ $\vec{\nabla} \times \vec{F} = 0$

$\therefore$ $U(r)$ exists, $\vec{F} = -\vec{\nabla} U$



(a) (b)

$(\vec{r}_1, \vec{r}_2)$

alternatively

$$(\vec{R}, \vec{r}) \longrightarrow \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$r = |\vec{r}_1 - \vec{r}_2| .$$

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r) .$$

Transforming coordinates to $(\vec{r}, \vec{R})$.

$$\boxed{M = m_1 + m_2}$$

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$$

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

reduced mass.

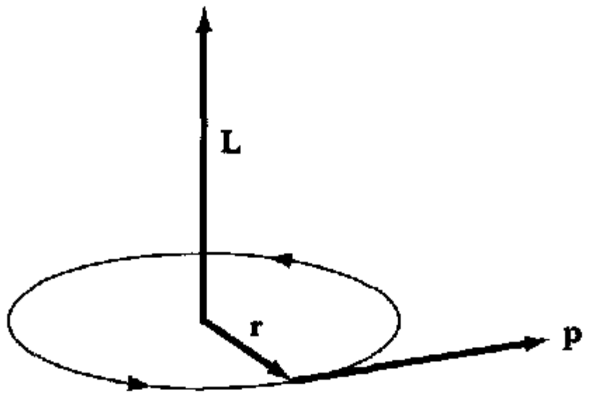
$\vec{R} = 0$, $\dot{\vec{R}} = 0$ CM frame.

$$\boxed{\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)} \longrightarrow$$

effective one-body problem with mass μ.

Conserved Quantities

- Energy is conserved
- $\mathcal{L}$ is spherically symmetric, θ, ϕ both cyclic. corresponding generalized momenta are conserved



$$\vec{F} = F(r)\,\hat{r}$$

Torque $\vec{N} = \vec{r} \times \vec{F} = 0$, Angular momentum is conserved.

direction of $\vec{L}$ is const

$\vec{L} = \vec{r} \times \vec{p}$ $\quad$ $\vec{L}$ is $\perp$ to $\vec{r}, \vec{p}$, plane containing $\vec{r}, \vec{p}$

$\rightarrow$ $\boxed{\text{motion is planar}}$

Can use 2d polar coordinates

$$\boxed{\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)}$$

θ is cyclic

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

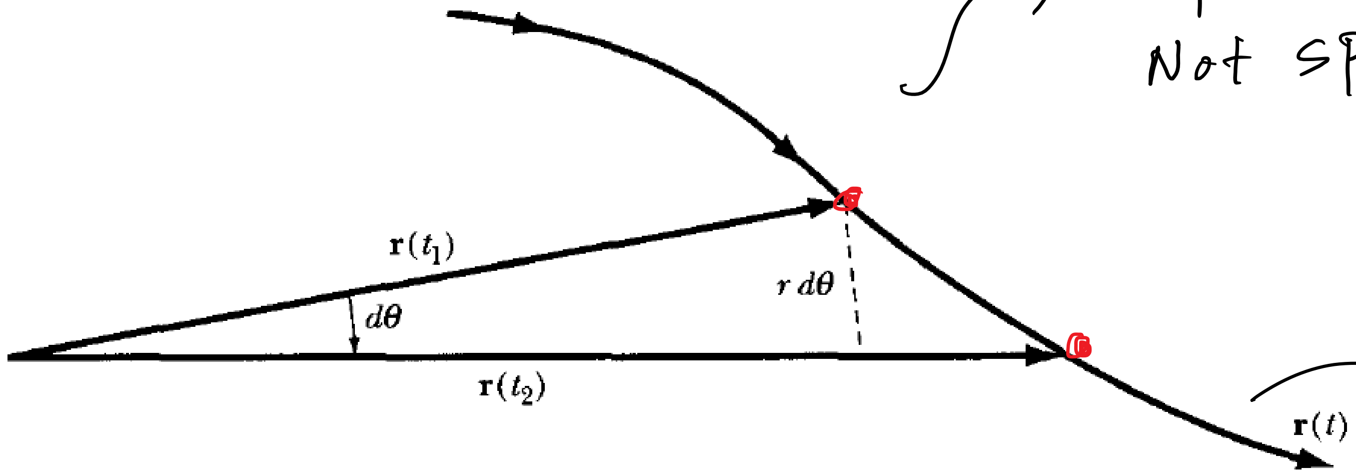
$$\boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const}}$$

Ang mom conservation

$$l = \mu r^2 \dot{\theta} = \text{const}$$

ang mom.

Kepler's 2nd Law
Not specific to $\frac{1}{r^2}$ force



trajectory.

· Geometrical interpretation

Area swept out by radius vector in time dt

$$dA = \frac{1}{2} r \cdot r\, d\theta = \frac{1}{2} r^2 d\theta$$

$$\begin{cases} l = \mu r^2 \dot{\theta} \\ \dot{\theta} = \frac{l}{\mu r^2} \end{cases}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2\mu} = \text{const}$$

$$\boxed{\frac{dA}{dt} = \text{const}}$$

Energy is conserved

$$E = T + U = \text{const}$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)$$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)}$$ Using $l = mr^2\dot{\theta}$

effectively 1-d problem.

Recall that a 1-d problem is in principle solvable completely

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r).$$

Using above to solve for $\dot{r}$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}} \quad \text{---} \ (*)$$

$$t = \pm\int \frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu^2 r^2}}}$$

$t = t(r)$

$\hookrightarrow$ invert to get $r(t)$

Our interest is to find the trajectory $r(\theta)$.

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \frac{\dot\theta}{\dot r}\,dr \qquad \left\{ \dot\theta = \frac{l}{\mu r^2} \right\}$$

$$= \frac{\frac{l}{\mu r^2}}{\dot r}\,dr \quad \text{—— } (**)$$

$\dot r \rightarrow$ can be obtained from eqn (*)

Integrating (* *)

$$\boxed{\theta(r) = \int \frac{\pm(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}} \quad \text{—— } (1)$$

$F(r) \propto r^n$

$n = 1, -2, -3$

expressible in terms of sin, cos fns.

Obs. Since l is const in time

$$l = mr^2\dot{\theta}$$

$\dot{\theta}$ cannot change sign.

$\theta(t)$ must monotonically increase or decrease with time.