Physics I

Lecture 20

Recall that a 1-d problem is in principle solvable completely

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r).$$

Using above to solve for $\dot{r}$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}} \quad \text{——} \; (*)$$

$$t = \pm\int \frac{dr}{\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

$$t = t(r)$$

invert to get $r(t)$

$$\ell = mr^2\dot{\theta}$$

$$\theta = \int \frac{\ell}{mr^2}\, dt$$

substitutes $r(t)$

Our interest is to find the trajectory $r(\theta)$.

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \qquad \left\{ \dot{\theta} = \frac{l}{\mu r^2} \right\}$$

$$= \frac{\frac{l}{\mu r^2}}{\dot{r}} dr \qquad \text{—— } (**)$$

$\dot{r}$ → can be obtained from eqn (*)

Integrating (**)

$$\boxed{\theta(r) = \int \frac{\pm(l/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}} \qquad \text{—— } (1)$$

$F(r) \propto r^n$

$n = 1, -2, -3$

expressible in terms of sin, cos fns.

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

E-L eqn.:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \qquad \theta : \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow \boxed{\mu r^2 \dot{\theta} = l} \quad — (1')$$

$$\boxed{\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U(r)}{\partial r} = F(r)} \quad — (1)$$

Change variable to $u = \frac{1}{r}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \quad — (2)$$

from (1') $\dot{\theta} = \frac{l}{\mu r^2} \longrightarrow \frac{du}{d\theta} = -\frac{1}{\cancel{r^2}}\frac{\mu \cancel{r^2}}{l}\dot{r} = -\frac{\mu}{l}\dot{r}$ — (3)

$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r} \quad \text{--- (3)}$$

Now $$\frac{d^2u}{d\theta^2} = -\frac{\mu}{l}\frac{d\dot{r}}{d\theta} = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right) = -\frac{\mu}{l\dot{\theta}}\ddot{r}$$

again substitute for $\dot{\theta}$ above

$$\boxed{\frac{d^2u}{d\theta^2} = -\frac{\mu}{l \cdot l} \cdot \mu r^2 \ddot{r} = -\frac{\mu^2}{l^2} r^2 \ddot{r}} \quad \text{--- (4)}$$

$\left.\dot{\theta} = \frac{l}{\mu r^2}\right\}$

$\dot{\theta}^2 = \frac{l^2}{\mu^2 r^4}$

from (4) $$\boxed{\ddot{r} = -\frac{l^2}{\mu^2} u^2 \frac{d^2u}{d\theta^2} \;;\; r\dot{\theta}^2 = \frac{l^2}{\mu^2} u^3} \quad \text{--- (5)}$$

E-L eqn. in r

$$\mu(\ddot{r} - r\dot{\theta}^2) = F(r)$$

from (5) we have $\ddot{r} = -\frac{l^2}{\mu^2} u^2 \frac{d^2u}{d\theta^2}$

and $r\dot{\theta}^2 = \frac{l^2}{\mu^2} u^3$.

substitute in E-L eqn.

$$-\frac{l^2}{\mu} u^2 \frac{d^2u}{d\theta^2} - \frac{l^2}{\mu} u^2 = F\left(\frac{1}{u}\right).$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2} \frac{1}{u^2} F\left(\frac{1}{u}\right)} \longrightarrow \text{path eqn.}$$

say

$$\left\{\begin{array}{l} F(r) = k/r^2 \\ F(1/u) = ku^2 \\ R.H.S = -\frac{\mu k}{l^2} \end{array}\right.$$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r)}$$

$r(\theta)$.

Notice that $l = 0$, eqn. blows up. But should we worry? $mr^2\dot{\theta} = 0$ $\theta = \text{const}$, st line through origin

Qualitative analysis of motion

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) .$$

Recall 1d problem

$\dot{r} = 0$ gives turning pts.

$$E - U(r) - \frac{\ell^2}{2\mu r^2} = 0 .$$

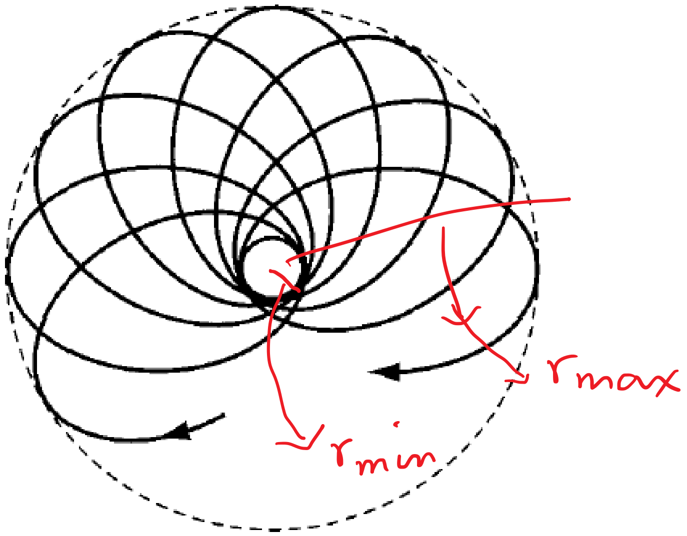
$E = U(x)$ turning pts.

$\downarrow$ motion is bounded & periodic.

$\downarrow$ In general possesses two roots r_{min}, r_{max}.

$$r_{min} \leq r \leq r_{max}$$

$\downarrow$ Can it be bounded but not periodic?



$r_{min} \leq r \leq r_{max}$

motion has to be confined to the annulus between r_{min} & r_{max}

If the motion is periodic, then orbit is closed

If the orbit does not close on itself after finite number of oscillations $\longrightarrow$ open

Recall $\theta(r)$

$$\theta(r) = \int \frac{\pm\left(\ell/r^2\right) dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{\ell/r^2}{\sqrt{2\mu\left(E - U - \ell^2/2\mu r^2\right)}}$$

If $U(r) \propto r^{n+1}$

closed, non circular path can result only for $n = -2$ or $+1$

motion is symmetric in time

→ path is closed if $\Delta\theta$ is a rational fraction of 2π

$$\Delta\theta = 2\pi \frac{m}{n}$$

, m, n are integers.

→ after n periods $\vec{r}$ made m complete revolutions.