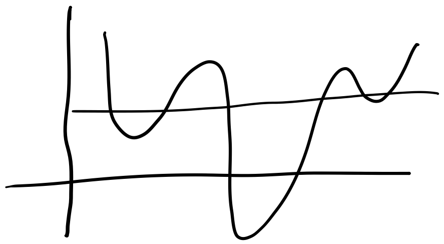
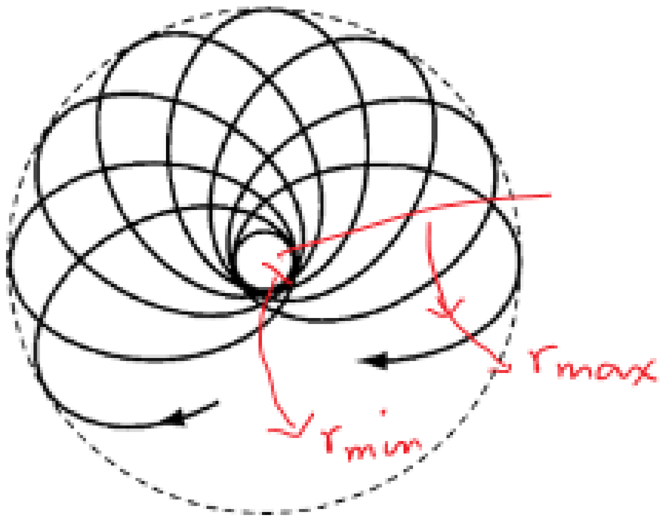
Physics I

Lecture 21

Recap

$E = U \rightarrow$ turning pts.





$r_{min} \leq r \leq r_{max}$

motion has to be confined to the annulus between r_{min} & r_{max}

If the motion is periodic, then orbit is closed $\rightarrow \Delta\theta = \frac{2\pi m}{n}$

If the orbit does not close on itself after finite number of oscillations $\rightarrow$ open

Effective potential

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{V(r) \equiv U_{eff}(r)}$$

$$\boxed{V(r) \equiv U(r) + \frac{\ell^2}{2\mu r^2}}$$

centrifugal potential energy

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r)$$

Let us specify $F(r) = -\frac{k}{r^2}$

$$U(r) = -\frac{k}{r}$$

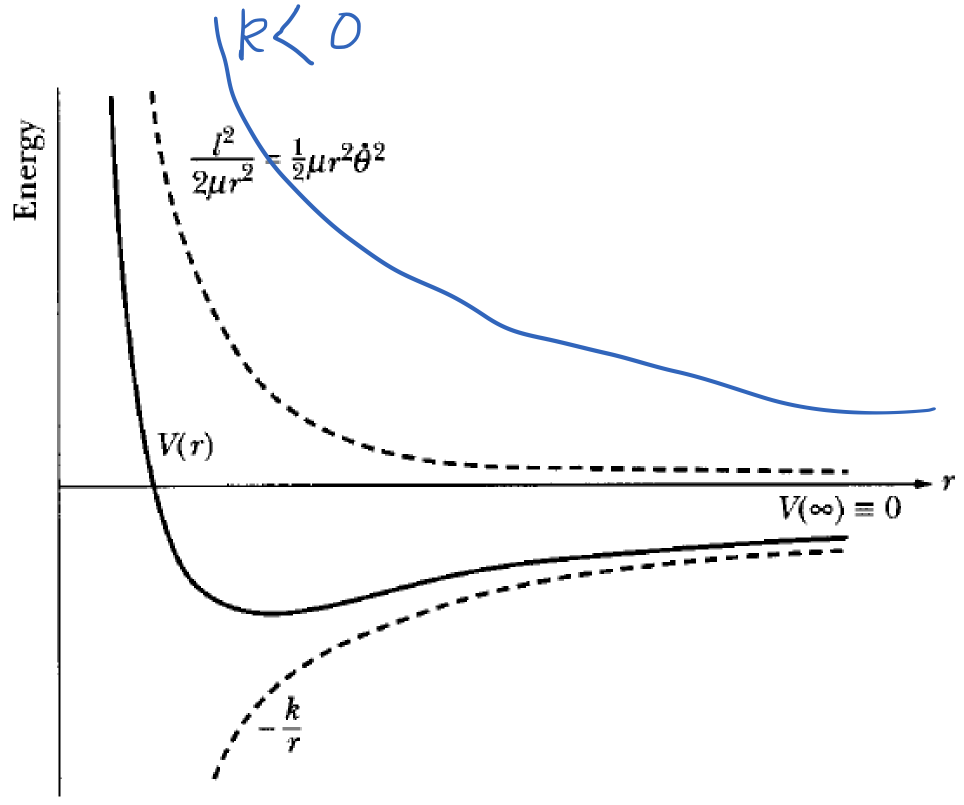
where we have taken $U(\infty) = 0$.

$$\boxed{V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}}$$

Recall : $E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + U(r)$

But $\mu r^2 \dot{\theta} = l$, $\dot{\theta} = \frac{l}{\mu r^2}$

$k < 0$



$$V(r) = \frac{l^2}{2\mu r^2} - \frac{k}{r}$$

iv) $E = E_1 \longrightarrow$ also unbounded

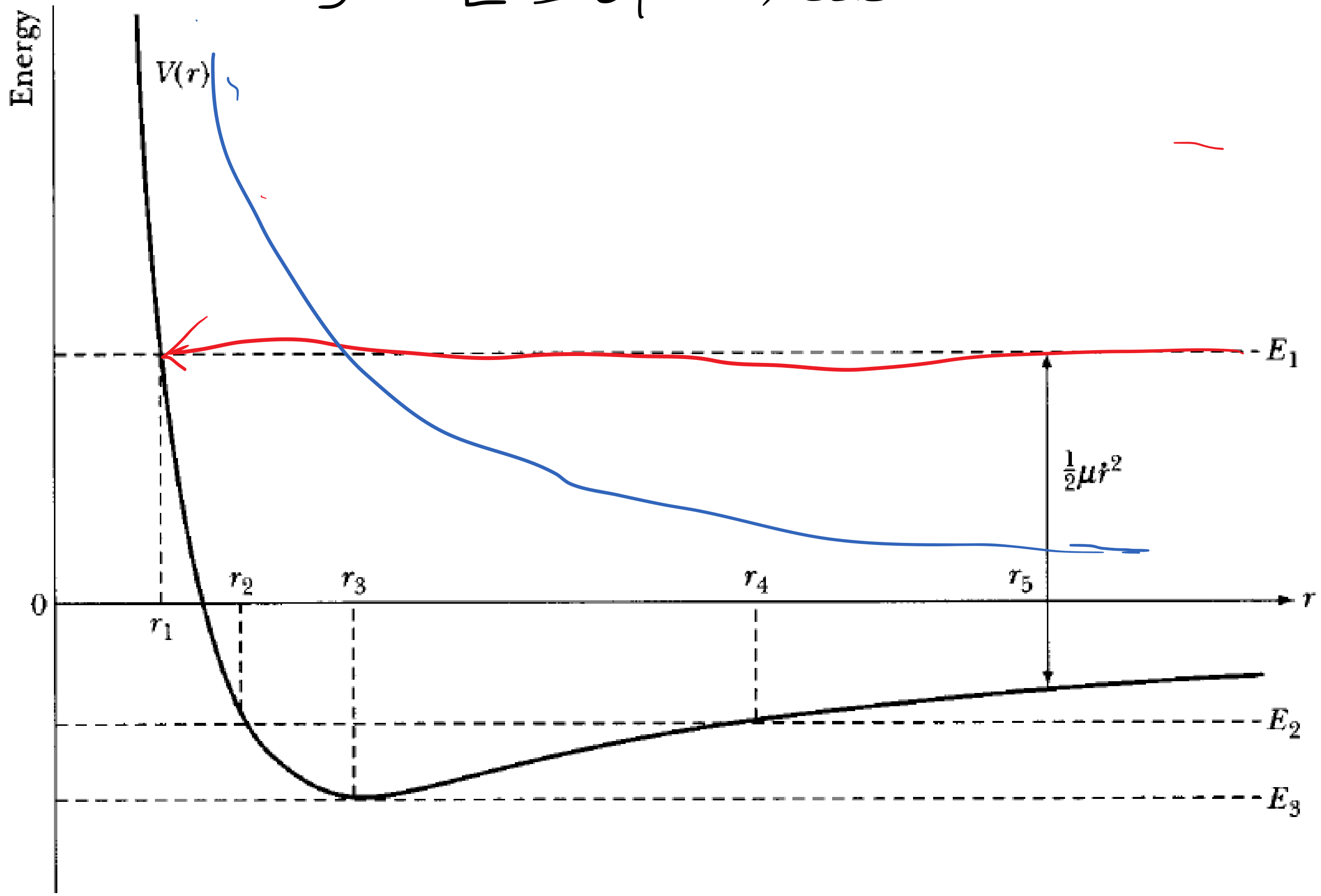
$$E = \frac{1}{2}\mu \dot{r}^2 + V(r)$$

If

$k < 0$

repulsive electrostatic potential

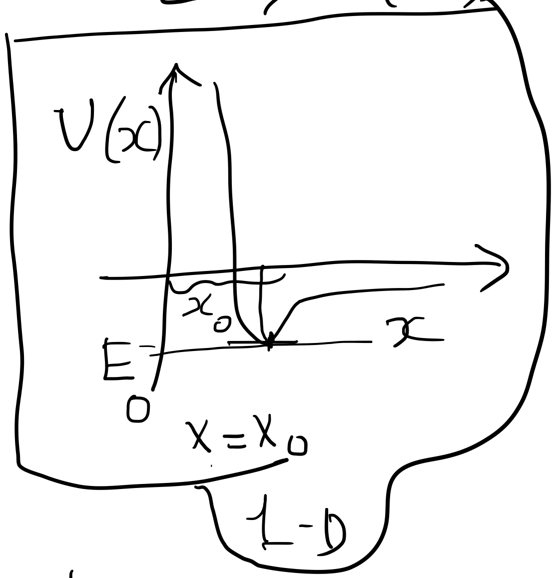
all energies unbounded motion



$E = 0$

Classically allowed

$E > V(r)$



i) $E = E_3$, E_3 minimum, $r = r_3$, $r = \text{const}$, circular orbit.

ii) $E = E_2$, $r_2 \leq r \leq r_4$, bounded

iii) $E = 0$, unbounded, one turning pt

Recall the path eqn. [will give $r(\theta)$]

$$u = \frac{1}{r}.$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2u^2} F\left(\frac{1}{u}\right).$$

$$F = -\frac{k}{r^2}$$

$$= -ku^2.$$

$$= -\frac{\mu}{l^2u^2}(-ku^2)$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}} \longrightarrow$$

harmonic oscillator with const force

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}.$$

↳ solving

$$\boxed{u = \frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_0)} \quad (1)$$

θ_0 gives the initial position θ, orientation of orbit in plane.

Let us take A to be positive, which can be always done

$$u = \frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_0) \quad \text{—} (1)$$

Determine turning points from (1) $[r_1, r_2]$

$$\frac{1}{r_1} = \frac{\mu k}{l^2} + A \quad \text{———} (2)$$

and $$\frac{1}{r_2} = \frac{\mu k}{l^2} - A \quad \text{—} (3)$$

If we have $A > \frac{\mu k}{l^2}$, there will be only 1 turning pt.

r must be +ve.

We will compare the turning pts, with solns of $E = V$

$$E = -\frac{k}{r} + \frac{l^2}{2\mu r^2}.$$ $\longrightarrow$ determines turning pts.

$E = V(r) = U_{eff}$.

$$\frac{l^2}{2\mu r^2} - \frac{k}{r} - E = 0$$

solns are:

$$\frac{1}{r_1} = \frac{\mu k}{l^2} + \left[\left(\frac{\mu k}{l^2}\right)^2 + \frac{2\mu E}{l^2}\right]^{1/2} \quad \text{— (4)}$$

$$\frac{1}{r_2} = \frac{\mu k}{l^2} - \left[\left(\frac{\mu k}{l^2}\right)^2 + \frac{2\mu E}{l^2}\right]^{1/2} \quad \text{— (5)}$$

Comparing 2, 3 & 4, 5, can determine A.

$$A^2 = \frac{\mu^2 k^2}{l^4} + \frac{2\mu E}{l^2} \quad — 6$$

Recall soln:

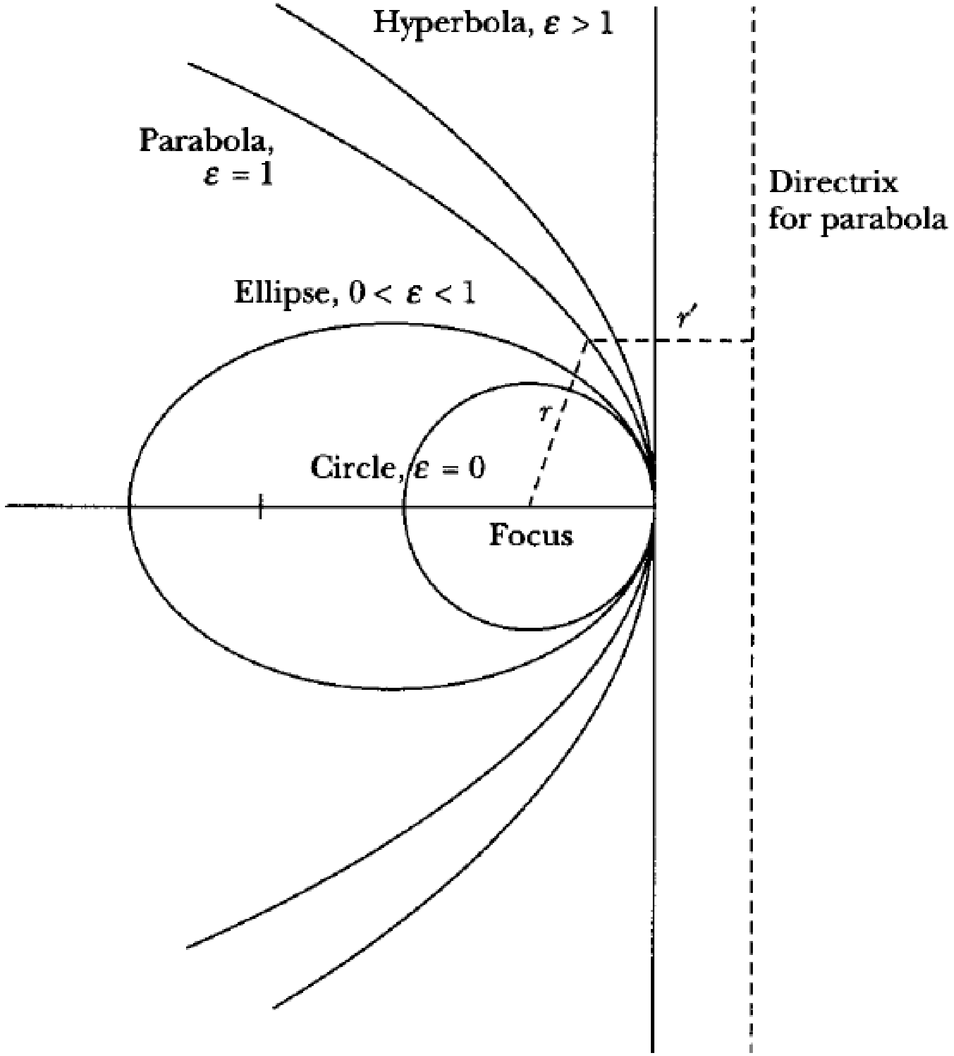
$$\frac{1}{r} = u = \frac{\mu k}{l^2} + A\cos\theta \quad — 1 \quad \text{Let } \theta_0 = 0$$

Let us define $\alpha \equiv \frac{l^2}{\mu k}$, $\varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$

Using these definitions, can rewrite 1

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta} \quad — 7$$

→ equation for general conic section in polar coordinates.



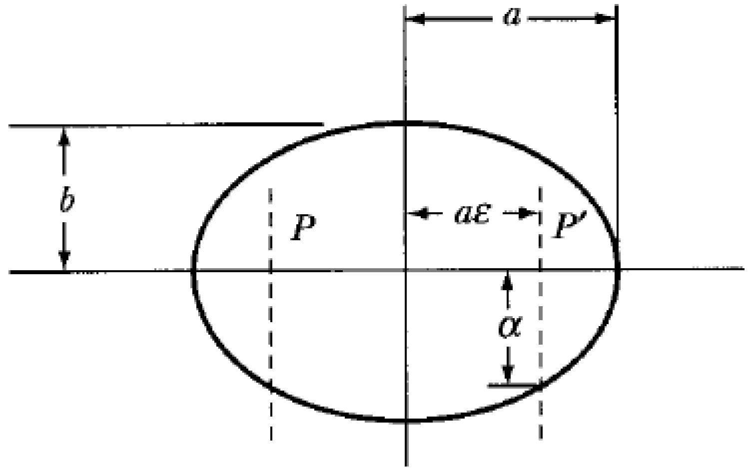
$\varepsilon > 1$, $E > 0$ hyperbola

$\varepsilon = 1$ $E = 0$ parabola

$0 < \varepsilon < 1$ $V_{min} < E < 0$

↳ ellipse

$\varepsilon = 0$ $E = V_{min}$ circle


a
b
P
aε
P′
α