Physics I

Lecture 22

Recap $\quad \vec{F} = -\frac{k}{r^2}\hat{r}$

solved path equation

eccentricity

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon\cos\theta}$$

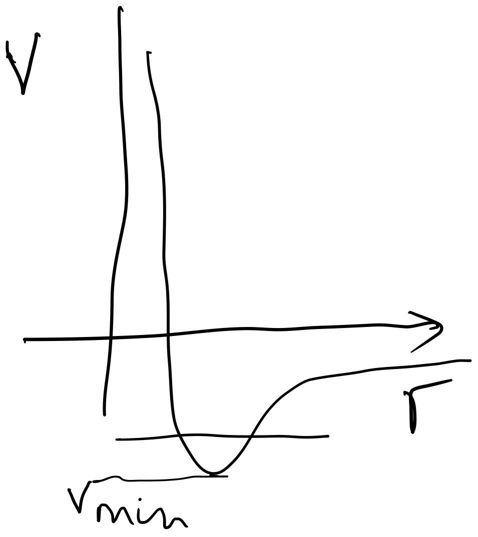
where $\alpha \equiv \frac{l^2}{\mu k}$

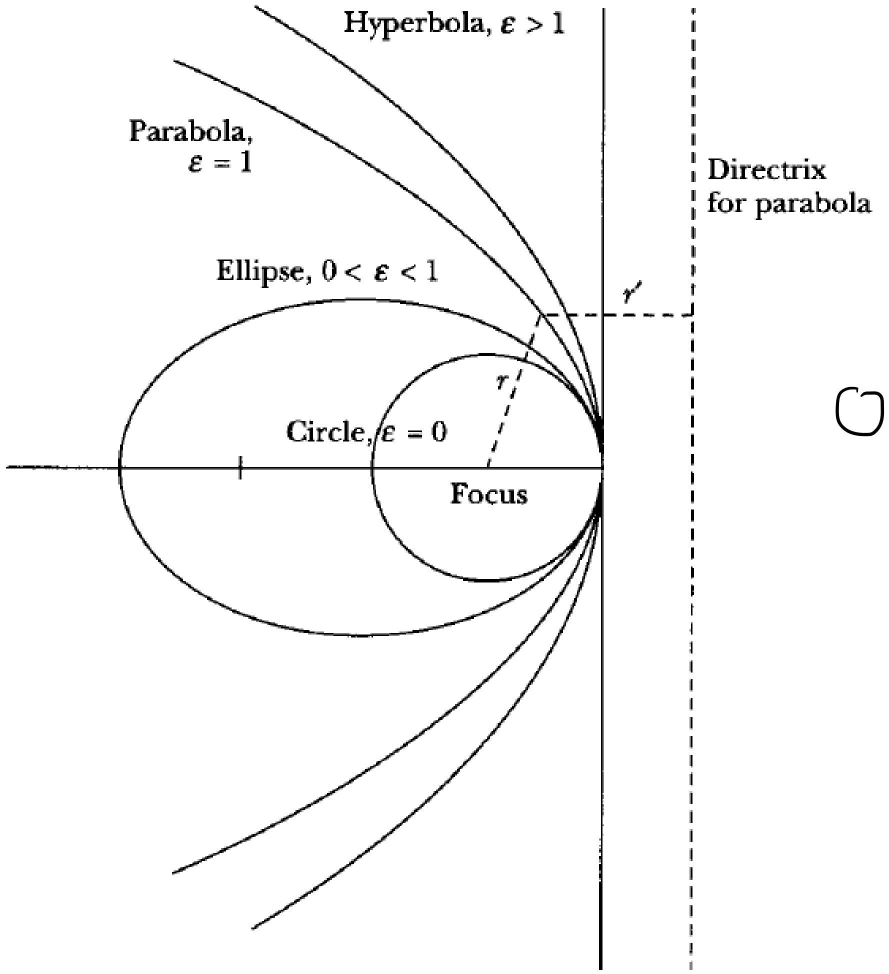
$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}}$.

conic section

familiar form recovered $\longrightarrow$ $x = r\cos\theta$

$y = r\sin\theta$.





$\varepsilon > 1$ $E > 0$ hyperbola

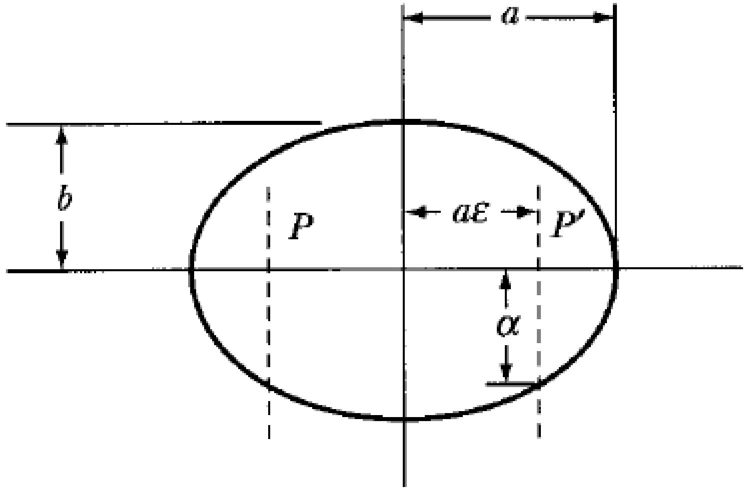
$\varepsilon = 1$, $E = 0$ parabola

$0 < \varepsilon < 1$, $V_{min} < E < 0$

$\downarrow$ ellipse

$\varepsilon = 0$, $E = V_{min}$

circular.



$r_{min} = a(1-\varepsilon) = \dfrac{\alpha}{1+\varepsilon}$.

$r_{max} = a(1+\varepsilon) = \dfrac{\alpha}{1-\varepsilon}$

pt corresponding to closest approach
perihelion

For planetary motion

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

Recall

$$\frac{dA}{dt} = \frac{l}{2\mu}$$: { Rate of sweeping out area }

Entire area of ellipse $\equiv$ swept out in one time period.

$$\int_0^{\tau} dt = \frac{2\mu}{l}\int_0^{A} dA .$$

$$\tau = \frac{2\mu}{l} A = \frac{2\mu}{l}\pi ab = \frac{2\mu}{l}\pi \frac{k}{2|E|}\cdot\frac{l}{\sqrt{2\mu|E|}}$$

$$\boxed{\tau = \pi k\sqrt{\frac{\mu}{2}}\,|E|^{-3/2}} \quad \text{——} \; (*)$$

$$\tau = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} \quad \text{---} \; (*)$$

$$\left\{ a = \frac{k}{2|E|} \right\}.$$

$$|E| = \frac{k}{2a}.$$

$$b = \sqrt{\alpha}\, a \qquad \alpha = \frac{l^2}{\mu k}.$$

→ Squaring $*$

$$\tau^2 = \pi^2 k^2 \frac{\mu}{2} |E|^{-3}.$$

$$\boxed{\tau^2 = \frac{4\pi^2 \mu}{k} a^3}$$

→ Kepler's Third Law with $m \to \mu$.

$$k = G m_1 m_2$$

$$\tau^2 = \frac{4\pi^2 \mu}{k} a^3 \qquad \left\{ \mu = \frac{m_1 m_2}{m_1 + m_2} \right\}$$

$$= \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

$$\boxed{\tau^2 \cong \frac{4\pi^2 a^3}{G m_2}} \quad m_1 \ll m_2 .$$

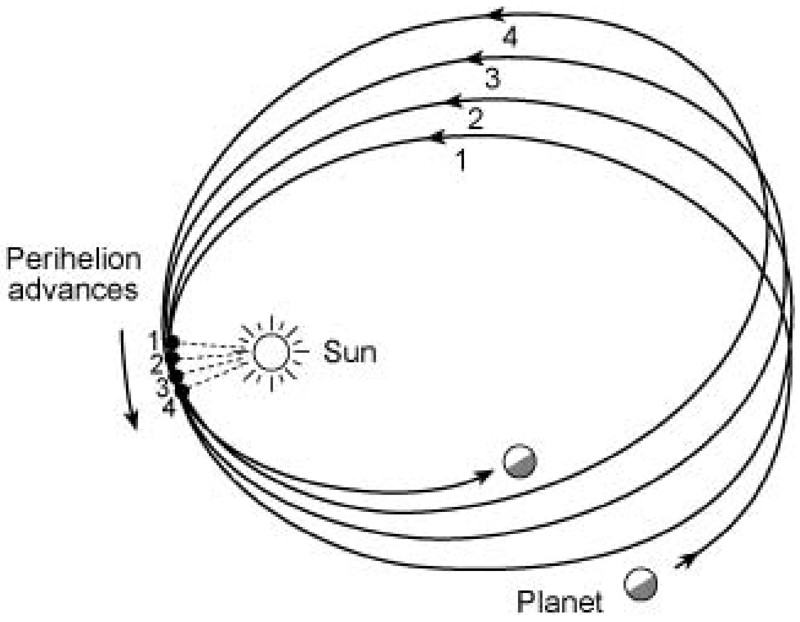
I. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

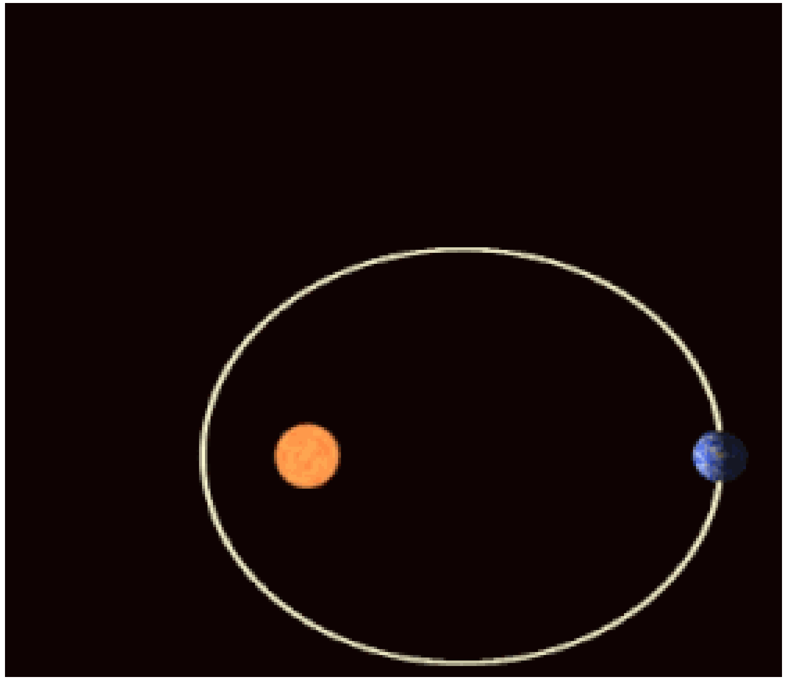
II. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

III. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

Actual orbits of planets are not strictly elliptical

→ nearby planets perturb the sun-planet gravitation field, ellipses dont come back to same point ⟶ perihelion precesion


4
3
2
1
Perihelion
advances
1
2
3
4
Sun
Planet



Mercury has largest perihelion shift

574'' arc sec / century

↳ all but 43'' / century could be explained by perturbations from other planets.

↳ this was explained by Einstein GR.

"effective correction ~~for~~ from GR" $\sim \frac{1}{r^4}$

Path eqn:

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)$$

~~Solve perturbatitively~~

here

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{\ell^2} + \frac{3GMu^2}{c^2}.$$

small term²

Conserved quantities in central force motion

E, l ✗

Laplace – Runge – Lenz vector:

$\vec{F}(r) = -\frac{k}{r^2}\hat{r}$, $\vec{L}$: angular momentum

$$\boxed{\vec{A} = \vec{P} \times \vec{L} - mk\hat{r}}$$

$E = \frac{1}{2}mv^2 - \frac{k}{r}$.

$\hookrightarrow \vec{A} \perp \vec{L}$; $\vec{P} \times \vec{L}$ and $\hat{r}$ are $\perp \vec{L}$

$\downarrow$ $\vec{A}$ lies in plane of motion

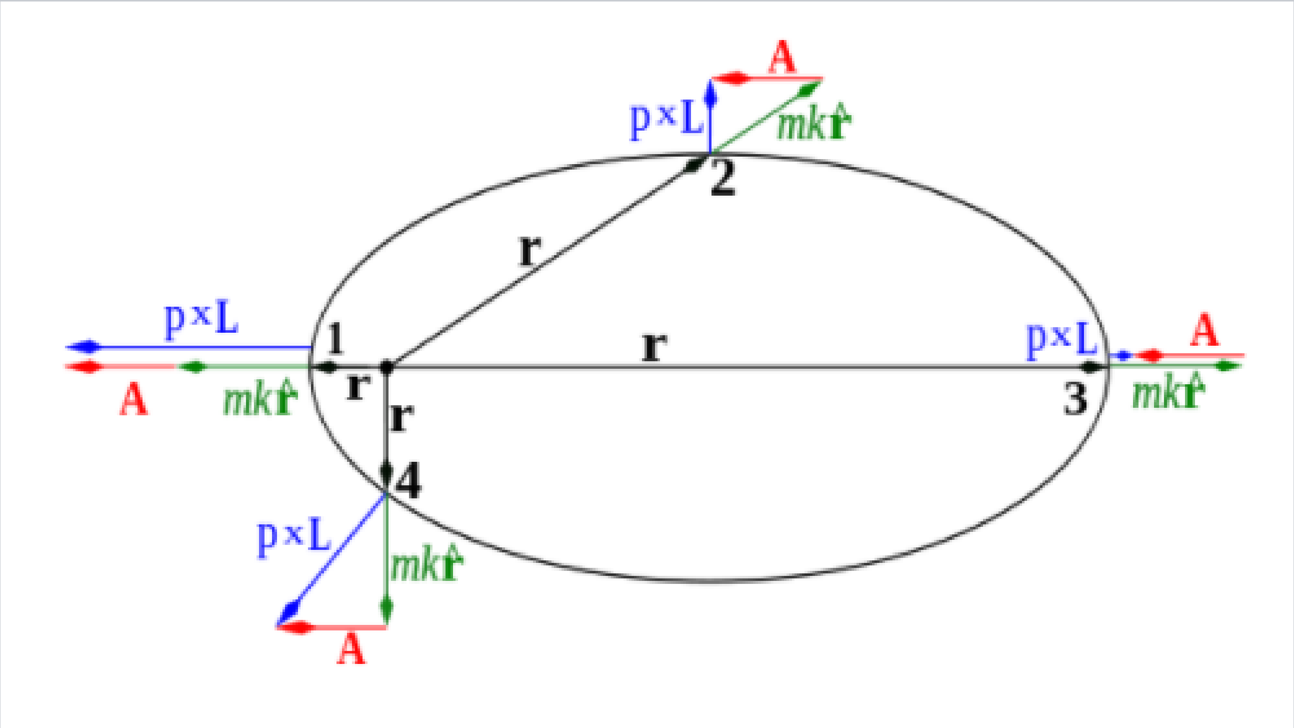


Figure 1: The LRL vector **A** (shown in red) at four points (labeled 1, 2, 3 and 4) on the elliptical orbit of a bound point particle moving under an inverse-square central force. The center of attraction is shown as a small black circle from which the position vectors (likewise black) emanate. The angular momentum vector **L** is perpendicular to the orbit. The coplanar vectors **p** × **L** and (*mk*/*r*)**r** are shown in blue and green, respectively; these variables are defined below. The vector **A** is constant in direction and magnitude.

Conservation

$$\vec{F} = \frac{d\vec{p}}{dt} = f(r)\frac{\vec{r}}{r} = f(r)\,\hat{r} \longrightarrow \text{central } f.$$

Want to show $\dfrac{d\vec{A}}{dt} = 0$ $\quad \dfrac{d}{dt}\left[\vec{p}\times\vec{L} - mk\hat{r}\right] = 0$

$$\frac{d}{dt}\left(\vec{p}\times\vec{L}\right) = \frac{d\vec{p}}{dt}\times\vec{L} \qquad \left(\because \frac{d\vec{L}}{dt} = 0\right).$$

$$= f(r)\,\hat{r}\times\left(\vec{r}\times m\frac{d\vec{r}}{dt}\right) \qquad \left[\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\right]$$

$$= f(r)\frac{m}{r}\left[\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) - r^2\frac{d\vec{r}}{dt}\right]$$

$$= f(r)\,\frac{m}{r}\left[\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) - r^2\,\frac{d\vec{r}}{dt}\right]$$

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\,\vec{r}\cdot\frac{d\vec{r}}{dt} = \frac{d}{dt}(r^2) = 2\,r\,\frac{dr}{dt}$$

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = -m f(r)\, r^2\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right]$$

$$= -m f(r)\, r^2\,\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = -m f(r)\, r^2\,\frac{d(\hat{r})}{dt}.$$

Now $f(r) = -\frac{k}{r^2}$

$$= \frac{mk}{\cancel{r^2}}\cdot\cancel{r^2}\,\frac{d\hat{r}}{dt} = \frac{d}{dt}(mk\hat{r}).$$

$$\frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{d}{dt}(mk\hat{r})$$

$$\boxed{\frac{d\vec{A}}{dt} = \frac{d}{dt}(\vec{p}\times\vec{L}) - \frac{d}{dt}(mk\hat{r}) = 0}$$

$$\frac{d\vec{A}}{dt} = 0$$