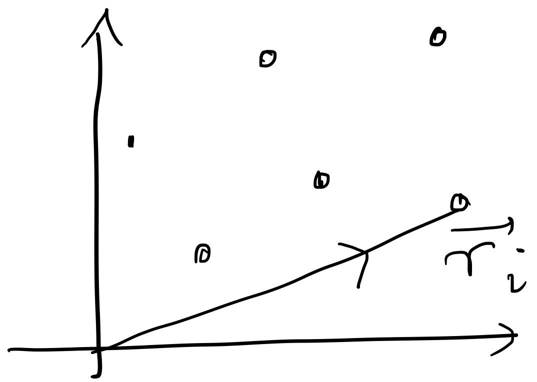
Physics I

Lecture 23

Many particle system dynamics

N particles



Newton's Law i^{th} particle

$$\dot{\vec{p}}_i = \sum_{\substack{j,\\ i\neq j}} \vec{F}_{ji} + \vec{F}_i^{(e)} \quad \text{—} (1)$$

Assume Newton's 3rd Law (weak form) not necessarily acting along line joining particles

$$\vec{F}_{ij} = -\vec{F}_{ji} \quad \text{—} (2)$$

Sum (1) over all particles.

$$\sum_{i=1}^{N} \dot{\vec{p}}_i = \underbrace{\sum_i \sum_{\substack{j\\ i\neq j}} \vec{F}_{ji}}_{=0} + \underbrace{\sum_i \vec{F}_i^{(e)}}_{\vec{F}^{(e)}} \quad \text{—} (3)$$

$\vec{F}^{(e)}$ = total ext force

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i$$

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)} \quad \text{—} \ 3$$

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \vec{F}^{(e)} \quad \text{—} \ 4$$

centre of mass coordinate.

Define $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \dfrac{\sum_i m_i \vec{r}_i}{M}$ — 5

4 reduces to

$$\boxed{M\frac{d^2\vec{R}}{dt^2} = \vec{F}^{(e)}} \quad \text{—} \ 6$$

purely internal forces have no effect on motion of CM.

Total linear momentum

$$\vec{P} = \sum_i m_i \frac{d\vec{r_i}}{dt} = M \frac{d\vec{R}}{dt} \qquad (7)$$

If $\vec{F}^{(e)} = 0$, total linear momentum is conserved

Angular Momentum

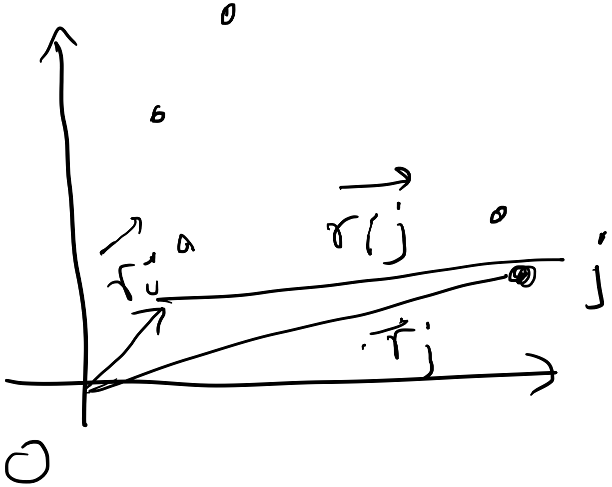
$$\vec{L}_{tot} = \sum_i (\vec{r}_i \times \vec{p}_i) = \vec{L}$$

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times F_i^{(e)} + \sum_i \sum_{\substack{j \\ i \neq j}} \vec{r}_i \times \vec{F}_{ji} \quad — 8$$

Last term can be considered as sum of pairs of the following form

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}}_{\text{vanishes}} \quad — 9$$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ in the direction of $\vec{F}_{ji}$



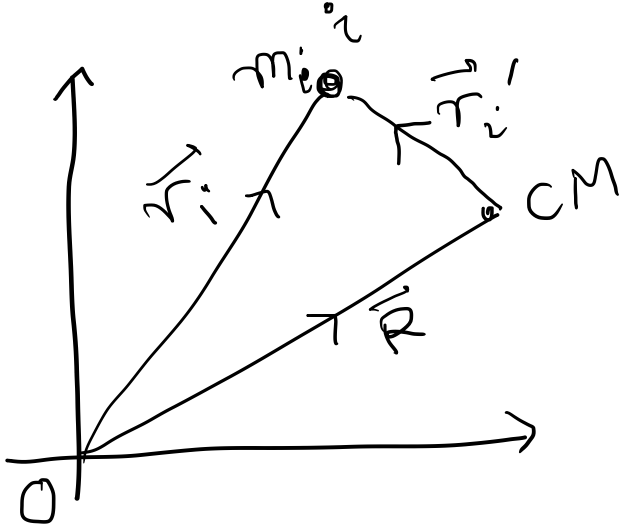
So if strong form of 3rd Law holds

$$(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = 0$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}^{(e)} = \vec{N}^{(e)}} \quad (10)$$

external torque

$$\boxed{\vec{N}^{(e)} = 0 \implies \vec{L} \text{ is conserved}}$$



Angular momentum about origin

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

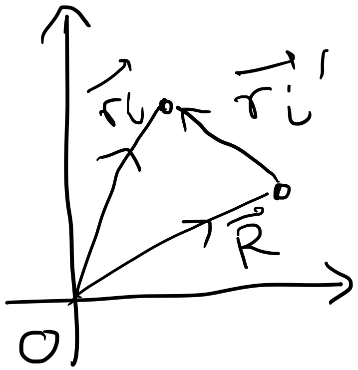
$$\boxed{\begin{aligned} \vec{V} &= \dot{\vec{R}} \\ \vec{v}_i' &= \dot{\vec{r}}_i' \\ \vec{v}_i &= \dot{\vec{r}}_i \end{aligned}}$$

$$\left.\begin{aligned} \vec{r}_i &= \vec{r}_i' + \vec{R} \\ \vec{v}_i &= \vec{v}_i' + \vec{V} \end{aligned}\right\} \text{—— } (11)$$

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{v}_i + \sum_i \vec{r}_i' \times m_i \vec{v}_i' + \sum_i \vec{r}_i' \times m_i \vec{V} + \sum_i \vec{R} \times m_i \vec{v}_i' \text{ —— } (12)$$

Rewriting (12)

$$\vec{L} = \sum_i \vec{R} \times m_i \vec{V} + \sum_i \vec{r_i}' \times m_i \vec{v_i}'$$



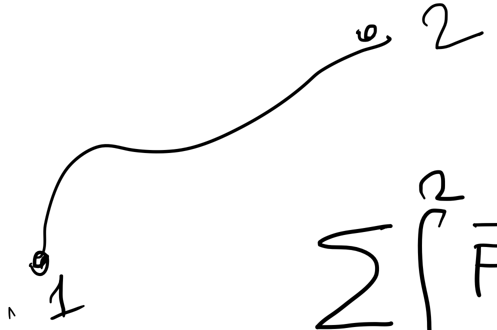
$$+ \underbrace{\left(\sum_i m_i \vec{r_i}'\right)}_{=0} \times \vec{V} + \vec{R} \times \frac{d}{dt}\underbrace{\left(\sum_i m_i \vec{r_i}'\right)}_{=0} .$$

$$\boxed{\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r_i}' \times \vec{p_i}'} \quad (13)$$

If the C.M is at rest w.r.t O, angular momentum will be independent of point of ref.

Energy

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i$$



$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{\substack{j \\ i \neq j}} \int_1^2 \vec{F}_{ji} \cdot d\vec{s}_i$$

Using eqns of motion

$$\sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 m \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \sum_i \int_1^2 d\left(\frac{1}{2} m v_i^2\right)$$

$$W_{12} = T_2 - T_1 \,, \quad \text{where} \quad T = \frac{1}{2} \sum_i m_i v_i^2$$

Making use of transfn. to CM coordinates.

$$T = \frac{1}{2} \sum_i m_i \left(\vec{v}_i' + \vec{V}\right) \cdot \left(\vec{v}_i' + \vec{V}\right).$$

$$= \frac{1}{2} \sum_i m_i V^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \vec{V} \cdot \frac{d}{dt}\underbrace{\left(\sum m_i \vec{r}_i'\right)}_{=0}$$

$$\boxed{T = \underbrace{\frac{1}{2} M V^2}_{\text{K.E of CM}} + \underbrace{\frac{1}{2} \sum_i m_i v_i'^2}_{\text{K.E of motion about the C.M.}}} \quad (13)$$

RHS -

$$\sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i \, .$$

If ext force conservative.

$$= \sum_i \int_1^2 -\vec{\nabla}_i U_i \cdot d\vec{s}_i = -\sum_i \int_1^2 dU_i = -\sum_i U_i \Big|_1^2 \, .$$

If internal forces also conservative.

$\vec{F}_{ij}$ can be derived from a potential U_{ij}

$$U_{ij} = U_{ij}\left(|\vec{r}_i - \vec{r}_j|\right) \longrightarrow \text{to satisfy } 3^{rd} \text{ Law}$$

$$\vec{F}_{ij} = -\vec{\nabla}_i U_{ij} = +\nabla_j U_{ij} = -\vec{F}_{ji} \qquad (14)$$

$$-\vec{F}_{ij} = \vec{\nabla}_i U_{ij}(\vec{r}_i - \vec{r}_j) = \underbrace{(\vec{r}_i - \vec{r}_j)}_{\text{in the direction of line joining two particles}} \underbrace{f}_{\text{scalar fn}}$$

scalar fn.

in the direction of line joining two particles.

Please fill in steps.

finally, we find that

Total potential energy

$$\boxed{U = \sum_i U_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} U_{ij}}$$

Consequence $\boxed{T + U \Rightarrow \text{conserved}}$