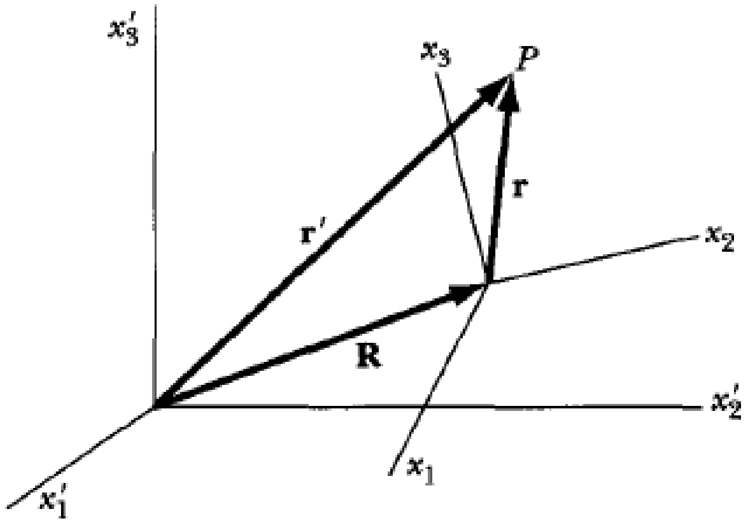
Physics I

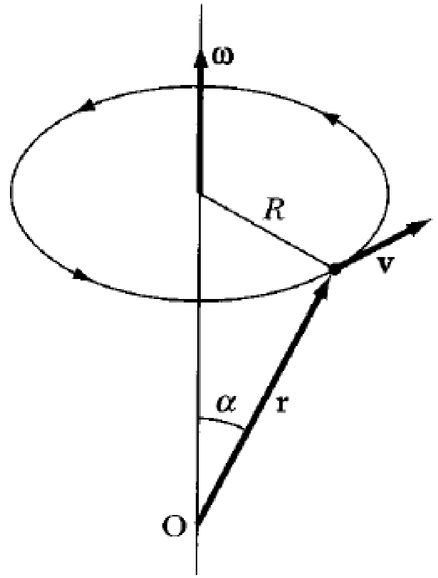
Lecture 27

Rotating Coordinate Systems



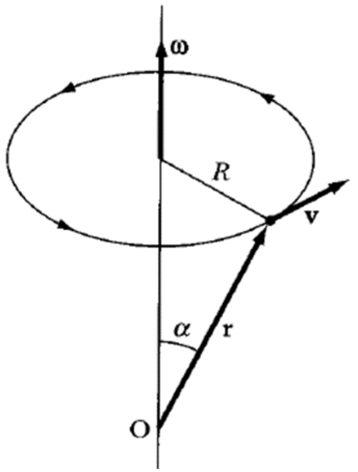
The x_i' are coordinates in the fixed system, and x_i are coordinates in the rotating system. The vector **R** locates the origin of the rotating system in the fixed system.

$$\boxed{\vec{r}\,' = \vec{R} + \vec{r}} \quad \text{— } (1)$$



Recall, we had learnt that a particle moving arbitrarily in space, can be considered , **at a given instant** to be moving in a **plane, circular path** about a given axis. An arbitrary infinitesimal displacement,(which can be a combination of translation and rotation) can always be represented by a “ pure rotation” about some axis called the instantaneous axis of rotation.

The line passing through the centre of the circle and perpendicular to the instantaneous direction of motion is called the instantaneous axis of rotation.



Rate of change of angular position $= \omega =$ angular velocity

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \quad \text{— (2)}$$

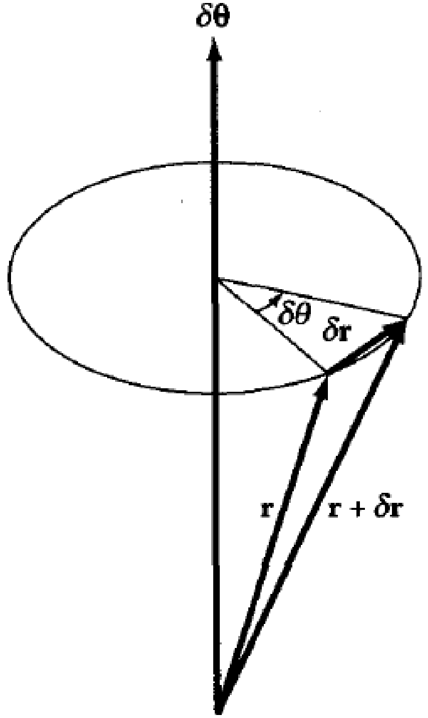
linear velocity $\vec{v} = \dot{\vec{r}}$

$$v = R\frac{d\theta}{dt} = R\omega \quad \text{— (3)}$$

$\vec{v} \perp \vec{r}$

$$v = r\omega \sin\alpha \quad \text{— (5)}$$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}} \quad \text{— (6)}$$



$$\boxed{\delta\vec{r} = \vec{\delta\theta} \times \vec{r}} \quad \text{--- (7)}$$

Getting back to our fixed vs rotating system

If x_i coordinate system undergoes infinitesimal rotation $\delta\theta$, for the motion P (at rest in x_i system)

$$\left(d\vec{r}\right)_{fixed} = d\vec{\theta} \times \vec{r} \quad \text{--- (8)}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \frac{d\vec{\theta}}{dt} \times \vec{r} \quad \text{--- (9)}$$

$\rightarrow$ essentially same as (6)

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega} \times \vec{r} \quad \text{--- (10)}$$ [P fixed in x_i system]

Now if the point P has velocity $\left(\frac{d\vec{r}}{dt}\right)_{rotating}$ w.r.t the x_i system, this must be added to $\vec{\omega} \times \vec{r}$ to obtain $\left(\frac{d\vec{r}}{dt}\right)_{fixed}$.

$$\boxed{\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{r}} \quad \text{— (11)}$$

Allthough we have derived (11) w.r.t $\vec{r}$, i.e the displacement vector, this holds <u>for any arbitrary vector $\vec{Q}$</u>

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{Q}} \quad \text{— (12)}$$

<u>In particular $\vec{Q} = \vec{\omega}$</u>

$$\left(\dot{\vec{\omega}}\right)_{fixed} = \left(\dot{\vec{\omega}}\right)_{rotating} + \underbrace{\vec{\omega}\times\vec{\omega}}_{=0} = \left(\dot{\vec{\omega}}\right)_{rotating} \quad \text{— (13)}$$

Let us seek transformation of velocities

$$\vec{r}\,' = \vec{R} + \vec{r}$$

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{fixed}} \quad \text{— } 14$$

Now using 12

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{\text{fixed}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{rotating}} + \vec{\omega} \times \vec{r} \quad \text{— } 15$$

Define

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} \equiv \vec{v}_f \equiv \dot{\vec{r}}_f \quad \text{---} (16a)$$

$$\left(\frac{d\vec{R}}{dt}\right)_{fixed} \equiv \vec{V} = \dot{\vec{R}}_f \quad \text{---} (16b).$$

$$\left(\frac{d\vec{r}}{dt}\right)_{rotating} = \vec{v}_r = \dot{\vec{r}}_r \quad \text{---} (16c).$$

Can rewrite (15) as:

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}} \quad \text{---} (17)$$

$\vec{v}_f$: vel. w.r.t fixed axis

$\vec{V}$: Linear vel of moving origin

$\vec{v}_r$: vel. w.r.t rotating axis

$\vec{\omega} \times \vec{r}$: vel. due to rotation of moving axis

- $\vec{F} = m\vec{a}$ valid only in inertial reference frame
  in this case $\longrightarrow$ fixed frame.

- $$\vec{F} = m\vec{a}_f = m\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} \quad \text{—— } (18)$$

Recall eqn. (17)

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}$$

Differentiating, we get

$$\left(\frac{d\vec{v}_f}{dt}\right)_{fixed} = \left(\frac{d\vec{V}}{dt}\right)_{fixed} + \left(\frac{d\vec{v}_r}{dt}\right)_{fixed} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} \quad \text{—— } (19)$$

Recall eqn. (12)

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q}$$

Define $\ddot{\vec{R}}_f = \left(\frac{d\vec{V}}{dt}\right)_{fixed}$ — (20)

$$\left(\frac{d\vec{v}_r}{dt}\right)_{fixed} = \left(\frac{d\vec{v}_r}{dt}\right)_{rot} + \vec{\omega} \times \vec{v}_r . \quad \text{— (21)}$$

$$= \vec{a}_r + \vec{\omega} \times \vec{v}_r .$$

$$\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{\omega} \times \vec{r} \quad \text{— (22)}$$

Putting it all together (18) becomes.

$$\boxed{\vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + m\dot{\vec{\omega}}\times\vec{r} + m(\vec{\omega}\times(\vec{\omega}\times\vec{r})) + 2m\vec{\omega}\times\vec{v}_r} \quad -(23)$$

To an observer in the rotating coordinate system the "effective" force on the particle is

$$\vec{F}_{eff} = m\vec{a}_r \quad -(24)$$

$$= \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_r . \quad -(25)$$

$-m\ddot{\vec{R}}_f \Rightarrow$ results from translational accln. of x_i system w.r.t x_i' system.

$-m(\dot{\vec{\omega}} \times \vec{r}) \Rightarrow$ results from rotational accln. of x_i system w.r.t x_i' system.

$-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \Rightarrow$ centrifugal force term, familiar $m\omega^2 r$ $\vec{\omega} \perp \vec{r}$, -ve sign indicates direction outward.

$-2m\vec{\omega} \times \vec{v}_r \Rightarrow$ Coriolis force.

$\vec{F} = m\vec{a_f}$ valid in inertial frame

$\vec{F}_{eff} = m\vec{a_r}$ (let $\vec{R}_f$ and $\vec{\omega}$ be zero for simplicity).

then

$\vec{F}_{eff} = m\vec{a_f}$ + (non-inertial terms)

centrifugal + Coriolis