Physics I

Lecture 30

Let us consider a rigid body composed of N particles of masses m_α, $\alpha = 1 \cdots N$.

$$\boxed{\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}} \quad \text{—} \quad (1)$$

inst. vel of α^{th} particle in fixed system

rigid body rotates with an instantaneous ang vel $\vec{\omega}$ about some pt fixed w.r.t body coordinate system (origin), and this pt. moves with linear vel $\vec{V}$ w.r.t fixed inertial coordinate system.

But the rigid body condn.

$$\boxed{\vec{v_r} = \left(\frac{d\vec{r}}{dt}\right)_{rotating} = 0} \rightarrow (2)$$

$\therefore$ from (1)

$$\boxed{\vec{V_\alpha} = \vec{V} + \vec{\omega} \times \vec{r_\alpha}} \quad (3)$$

K.E of α^{th} particle

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 \quad \text{--- (4)}$$

Total K.E (from (3))

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \left(\vec{V} + \vec{\omega}\times\vec{r}_\alpha\right)^2 \quad \text{--- (5)}$$

valid for arbitrary choice of origin

$$= \frac{1}{2}\sum_\alpha m_\alpha V^2 + \sum_\alpha m_\alpha \vec{V}\cdot\left(\vec{\omega}\times\vec{r}_\alpha\right) + \frac{1}{2}\sum_\alpha m_\alpha \left(\vec{\omega}\times\vec{r}_\alpha\right)^2 \quad \text{--- (6)}$$

Specializing to C.M as origin

$$2^{nd} \text{ term} = \vec{V}\cdot\vec{\omega}\times\sum_\alpha \underbrace{m_\alpha \vec{r}_\alpha}_{=0} \quad \text{--- (7)}$$

$$T = T_{trans} + T_{rot} \quad (8)$$

$$\boxed{\begin{aligned} T_{trans} &= \frac{1}{2}\sum_{\alpha} m_{\alpha} V^2 = \frac{1}{2} M V^2 \\ T_{rot} &= \frac{1}{2}\sum_{\alpha} m_{\alpha} (\vec{\omega} \times \vec{r}_{\alpha})^2 \end{aligned}} \quad (9)$$

Using the identity $(\vec{A} \times \vec{B})^2 = A^2 B^2 - (\vec{A} \cdot \vec{B})^2$.

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_{\alpha} \left[\omega^2 r_{\alpha}^2 - (\vec{\omega} \cdot \vec{r}_{\alpha})^2\right] \quad (10)$$

Express T_{rot} in components ω_i and $r_{\alpha i}$ of $\vec{\omega}$ and $\vec{r}_\alpha$: $\vec{r}_\alpha = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$

$r_{\alpha,i} \equiv x_{\alpha,i}$ in body system.

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k x_{\alpha,k}^2\right) - \left(\sum_i \omega_i x_{\alpha,i}\right)\left(\sum_j \omega_j x_{\alpha,j}\right)\right] \quad \text{--- (11)}$$

can write $\omega_i = \sum_j \omega_j \delta_{ij}$, where $\delta_{ij} = 0 \;\; i \neq j$
$= 1 \;\; i = j$

$$T_{rot} = \frac{1}{2}\sum_{\alpha}\sum_{i,j} m_\alpha \left[\omega_i \omega_j \delta_{ij}\left(\sum_k x_{\alpha,k}^2\right) - \omega_i \omega_j x_{\alpha,i} x_{\alpha,j}\right] \quad \text{(12)}$$

$$= \frac{1}{2}\sum_{i,j} \omega_i \omega_j \sum_{\alpha} m_\alpha \left(\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right) \quad \text{(13)}$$

Define

$$\boxed{I_{ij} = \sum_{\alpha} m_\alpha \left(\delta_{ij}\sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right)} \quad \text{(14)}$$

Now we have

$$\boxed{T_{tot} = \frac{1}{2} \sum_{i,j} \underbrace{I_{ij}} \omega_i \omega_j} \quad (15)$$

Moment of inertia tensor.

$\{I\} \rightarrow$ matrix.

In restricted form

$$T_{rot} = \frac{1}{2} I \omega^2 \quad (16)$$

$$\{I\} \doteq \begin{bmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,1} x_{\alpha,2} & -\sum m_\alpha x_{\alpha,1} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha x_{\alpha,2} x_{\alpha,3} \\ -\sum_\alpha m_\alpha x_{\alpha,3} x_{\alpha,1} & -\sum m_\alpha x_{\alpha,3} x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{bmatrix} \tag{17}$$

can be written in terms of

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha).$$

$$\{I\} = \begin{Bmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum m_\alpha x_\alpha y_\alpha & -\sum m_\alpha x_\alpha z_\alpha \\ -\sum m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{Bmatrix} \qquad (18)$$

$\boxed{I_{ij} = I_{ji}}$

symmetric

Diagonal elements $\Rightarrow$ Moments of inertia (I_{11}, I_{22}, I_{33}).

-ve of Off diagonal elements $\Rightarrow$ products of inertia

Angular momentum

w.r.t some pt. O fixed in body coordinate system

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \quad \text{--- (19)}$$

$$\vec{p}_\alpha = m_\alpha \vec{v}_\alpha = m_\alpha (\vec{\omega} \times \vec{r}_\alpha).$$

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha).$$

$$= \sum_\alpha m_\alpha \left[r_\alpha^2 \vec{\omega} - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega}) \right] \text{--- (20)}$$

$$L_i = \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j}\,\omega_j \right)$$

$$= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j}\, \omega_j \right)$$

$$= \sum_j \omega_j \sum_\alpha \underbrace{m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i}\, x_{\alpha,j} \right)}_{I_{ij}}$$

$$\boxed{L_i = \sum_j I_{ij}\,\omega_j} \quad (21)$$

$$\left\{ \text{special case} \quad \vec{L} = I\,\vec{\omega} \right\}$$