Physics I

Lecture 31

$$I_{ij} = \sum_{\alpha} m_{\alpha} \left(\delta_{ij} \sum_{k} x_{\alpha,k}^{2} - x_{\alpha,i} x_{\alpha,j} \right)] \quad \text{--- } (1)$$

$$T_{rot} = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \quad \text{--- } (2)$$

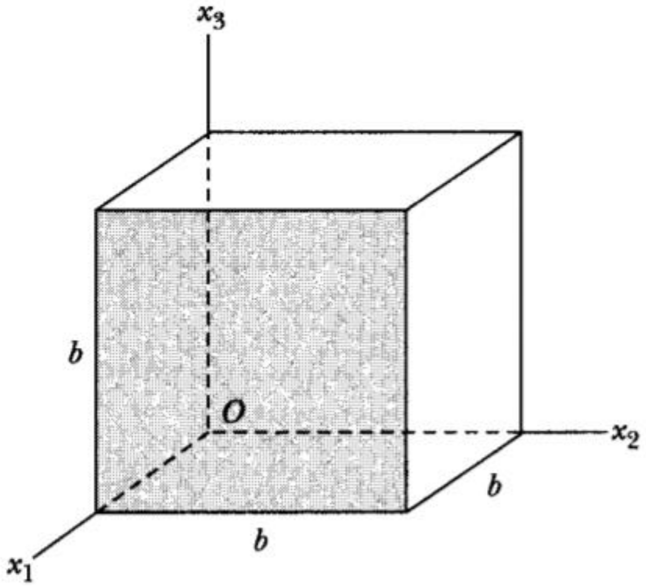
$$L_i = \sum_{j} I_{ij} \omega_j \quad \text{--- } (3)$$

$$T = \frac{1}{2} \sum_{i} L_i \omega_i = \frac{1}{2} \vec{\omega} \cdot \vec{L} \quad \text{--- } (4)$$

Continuum

$$I_{ij} = \int_{V} \rho(\vec{r}) \left(\delta_{ij} \sum x_k^2 - x_i x_j \right) dv \quad \text{--- } (5)$$

$$dv = dx_1 dx_2 dx_3 .$$



homogeneous cube of density ρ, mass M, side b.

$$I_{ij} = \int_V dv\, \rho \left[\delta_{ij} \sum_k x_k^2 - x_i x_j \right]$$

$$I_{11} = \rho \int_0^b dx_1 \int_0^b dx_2 \int_0^b dx_3 \, (x_2^2 + x_3^2)$$

$$= \frac{2}{3} \rho b^5 = \frac{2}{3} M b^2 \quad \text{--- (6)}$$

Let

$$\boxed{Mb^2 = \beta}$$

$$I_{11} = I_{22} = I_{33} = \frac{2}{3}\beta \quad \text{--- (7)}$$

All the off diagonal elements are equal too.

$$I_{12} = -\rho \int_0^b x_1 \, dx_1 \int_0^b x_2 \, dx_2 \int_0^b dx_3 .$$

$$= -\frac{1}{4}\rho b^5 = -\frac{1}{4} M b^2 . \quad \text{— (8)}$$

$$I_{12} = I_{13} = I_{23} = -\frac{1}{4}\beta \quad \text{— (9)}$$

$$\{I\} = \begin{Bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{Bmatrix} \quad 10$$

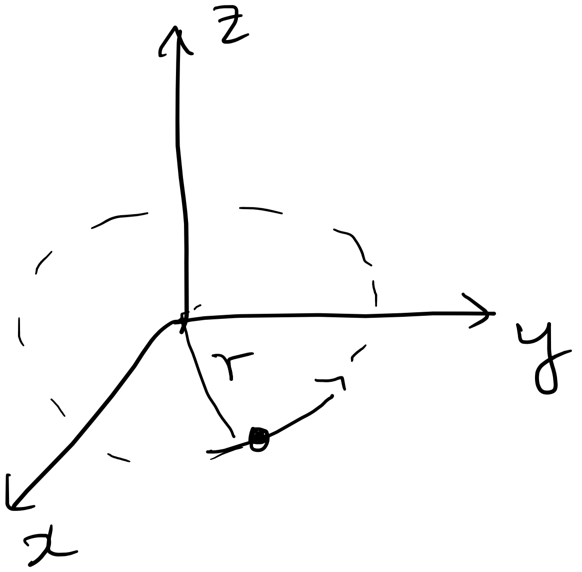
$$L_i = \sum_j I_{ij} \omega_j \quad \text{--- (3)}$$

If the inertia tensor has non-vanishing off diagonal elements; then say $\vec{\omega} = (\omega_1, 0, 0)$.

$\vec{L}$ will have components in all directions. $\{L_1, L_2, L_3\}$

$$\boxed{\text{Angular momentum in general does not have same direction as ang. vel.}}$$

Example 1 (point mass in x-y plane)



$\downarrow$ travelling in a circle of radius r, with freq. $\vec{\omega} = (0, 0, \omega)$.

$$x^2 + y^2 = r^2, \quad z = 0.$$

Using (3)

$$L_i = \sum_j I_{ij} \omega_j$$

$$L_x = I_{xz} \omega_z, \quad L_y = I_{yz} \omega_z, \quad L_z = I_{zz} \omega_z$$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right).$$

$$I_{xz} = I_{yz} = 0 \qquad I_{zz} = mr^2$$

$$\boxed{L_z = mr^2\omega}$$

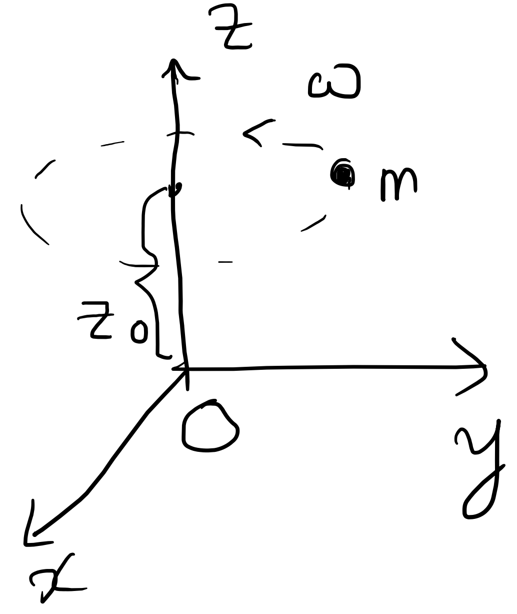
$L_x, L_y = 0$

$$\boxed{\vec{L} = mr^2\vec{\omega}}$$

$\vec{L}, \vec{\omega}$ in the same direction

Example 2. Point mass in space

$\vec{\omega} = (0, 0, \omega)$, $z = z_0$, $r^2 = x^2 + y^2$



$\vec{L}$ w.r.t O

$I_{xz} = -m x z_0$ $I_{yz} = -m y z_0$

$I_{zz} = m r^2$.

$\vec{L} = m\omega(-x z_0, -y z_0, r^2)$.

$L_x, L_y \neq 0$

$\vec{L}$ and $\vec{\omega}$ are not in the same direction.

Look through worked out examples 11.4 in Marion Thornton.

Principal Axes of Inertia

$$T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \quad — 1$$

$$L_i = \sum_j I_{ij} \omega_j \quad — 2$$

If I tensor had only diagonal elements

$$I_{ij} = I_i \delta_{ij} \Rightarrow 3 \qquad \{I\} = \begin{Bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{Bmatrix} — 4$$

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 \quad \text{—} \quad (5)$$

$$L_i = \sum_j \delta_{ij} I_i \omega_j = I_i \omega_i \quad \text{—} \quad (6)$$

↳ find a set of body axes in which the products of inertia vanish

⟶ Principal axes of inertia.

$$\left.\begin{aligned} L_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3. \\ L_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\} \text{— (7)}$$

$\vec{L} = I\vec{\omega}$ — (8) body rotating about a principal axis

Combining (7), (8)

$$\left.\begin{aligned} (I_{11}-I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22}-I)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33}-I)\omega_3 &= 0 \end{aligned}\right\} \text{— (9)}$$

Eqns (9) will have non trivial solns provided

$$\begin{vmatrix} (I_{11}-I) & I_{12} & I_{13} \\ I_{21} & (I_{22}-I) & I_{23} \\ I_{31} & I_{32} & (I_{33}-I) \end{vmatrix} = 0 \quad \text{— (1)}$$

→ secular eqn is cubic, each root is called a principal moment of inertia (I_1, I_2, I_3).

→ directions are determined by eigenvectors